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Diffusion on Rapidly-Varying Surfaces

by

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Thesis

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DECLARATIONS

All work in this thesis unless otherwise stated is the sole work of the author.

ABSTRACT

Lateral diffusion of molecules on surfaces plays a very important role in various biological processes, including lipid transport across the cell membrane, synaptic transmission and other phenomena such as exo- and endocytosis, signal transduction, chemotaxis and cell growth. In many cases, the surfaces can possess spatial inhomogeneities and/or be rapidly changing shape. In this thesis we consider the problem of lateral diffusion on quasi-planar surfaces, which are fluctuating according to various models. Using homogenisation theory, we show that, under the reasonable assumption of well separated scales, the lateral diffusion process can be well-approximated by a Brownian motion on the plane with constant diffusion coefficient D. The diffusion coefficient D will depend in a complicated way on the different properties of the surface, such as the average excess surface area, and for biologically motivated models, the bending stress and surface tension.

We consider three classes of surface fluctuation models. The first case we consider is a periodic fluctuation model, where the surface is time-independent possessing rapid, periodic fluctuations. Using classical homogenisation techniques we obtain an expression for D for a particle diffusing on such a surface and are able to study the various properties of D. Although D will not have a closed-form expression in general, we identify a large class of two-dimensional surfaces for which the effective diffusion coefficient has an explicit form which depends only on the excess surface area.

The second model we consider is a static, stationary random field model, where the surface is given by a rapidly fluctuating, random field with stationary, ergodic fluctuations. Under appropriate assumptions, we are also able to prove a homogenisation result for lateral diffusion on such a surface and prove results analogous to those for the first model.

Generalising the thermally-excited Helfrich-elastic membrane model, the third case we consider is a fluctuating surface having both rapid spatial and temporal fluctuations. The effective diffusion coefficient will depend on the relative scales of the spatial and temporal fluctuations. For different scaling regimes, we prove the existence of a macroscopic limit in each case.

In each of the cases, the theoretical results are supplemented with numerical experiments which highlight the theory as well as explore scenarios not covered by theory.

Chapter 1

INTRODUCTION

Diffusive processes are ubiquitous in physics, chemistry and biology (see [Crank, 1979; Berg, 1993; Van Kampen, 2007]). In biology, diffusion plays a fundamental role in many processes occurring at the cellular and sub-cellular level, and is one of the basic mechanisms for intracellular transport [Bressloff and Newby, 2013]. Diffusion not only occurs within the cell, but can also occur along the cell membrane. This lateral diffusion of molecules along the surface of cells also plays a key role in various cellular processes. The lipid molecules and integral membrane proteins which constitute the cell membrane themselves undergo lateral diffusion as a result of thermal agitation, [Almeida and Vaz, 1995]. Lateral diffusion of postsynaptic membrane proteins between synapses is known to play a fundamental part in synaptic transmission, [Borgdorff et al., 2002; Ashby et al., 2006]. Other phenomena in which lateral diffusion over biological interfaces is involved include vision [Poo et al., 1974], exo- and endocytosis, signal transduction, chemotaxis and cell growth (see [Sbalzarini et al., 2006] and [Almeida and Vaz, 1995]).

Experimental techniques such as single particle tracking, [Saxton and Jacobson, 1997], fluorescence recovery after photobleaching (FRAP), [Axelrod et al., 1976] and nuclear magnetic resonance (NMR), [Lindblom and Orädd, 1994] have made it possible to accurately measure displacement in a laboratory fixed plane of molecules diffusing laterally on the surface, and thus to measure the macroscopic diffusion coefficient D of the trajectory of the diffusive process, projected into the plane.

Biological interfaces, however, are not typically flat. Indeed, many membranes will exhibit a non-zero curvature which is induced by the natural spontaneous curvature of the constituent lipids [Seifert, 1997]. They may also be rough, or possess some spatial microstructure. Moreover, the shape of the membrane is changing in time due to thermal fluctuations and possibly also non-thermal fluctuations induced by active membrane proteins on the surface [Gov, 2004].

The geometry of the membrane will cause the measured macroscopic diffusion coefficient D to be significantly different from the molecular diffusion coefficient D_0 of the diffusing protein on the surface itself. The relationship between the molecular diffusion coefficient and the measured diffusion coefficient has been widely studied for different types of biomembrane. Previous work such as [Gustafsson and Halle, 1997; Naji and Brown, 2007; Halle and Gustafsson, 1997] and [Sbalzarini et al., 2006] focus on the problem of lateral diffusion of a particle on a static membrane. Various estimates for D in terms of the surface fluctuation were derived, most notably the effective medium approximation and area scaling approximation [Gustafsson and Halle, 1997] and [Sokolov, 1987], and [Naji and Brown, 2007]. Other studies such as [Reister and Seifert, 2007; Reister-Gottfried et al., 2007, 2010] have focussed on the problem of diffusion on a thermally excited biomembrane fluctuating in a hydrodynamic medium, and derived expressions for the effective diffusion coefficient as a function of surface parameters such as bending rigidity, surface tension and fluid viscosity.

The common factor in these models is the presence of small length and time scales in the resulting evolution equations, which enter due to spatial surface microstructure, or due to rapid temporal fluctuations of the surface or possibly both. The objective of this thesis is to investigate the macroscopic behaviour of a laterally diffusive process on surfaces possessing multiple space and time scales using a single, unified mathematical approach. By doing so we provide rigorous justification for some existing approximations advocated in the literature, clearly explaining the parametric regimes in which they apply, and we develop a systematic methodology which can be used to study other similar problems. Under the assumption that the slow and fast scales are well-separated it is possible to show that the diffusion process can be approximated by a constant-coefficient diffusion process on the plane, independent of the small scale, but which accounts for the macroscopic effects of the fine spatial structure and rapid fluctuations. We use the classical methods of averaging and homogenisation (e.g. [Bensoussan et al., 1978; Pavliotis and Stuart, 2008]) and derive explicit expressions for the coefficients of the macroscopic process in terms of averages with respect to a relevant measure reflecting the rapid fluctuations, and involving solution of the auxiliary *cell equation* in the case of homogenisation. Although these coefficients will not have a closed form in general, they can be computed numerically, accurately and efficiently without having to simulate effects at the microscopic level, and they are amenable to analysis in various parameter regimes of interest.

The use of multiscale methods to study lateral diffusion on membranes has been considered before, with varying degrees of rigour. In [Gustafsson and Halle, 1997], the authors derive the correct macroscopic diffusion coefficient for a particle diffusing on a surface with periodic spatial fluctuations basing their result on [Jackson and Coriell, 1963] and [Festa and d'Agliano, 1978], who consider the analogous situation of diffusion in a periodic potential. Under the assumption of symmetry in the spatial fluctuations the authors then proceed to derive variational bounds for the effective diffusion, and provide heuristic arguments for a number of other, tighter approximations. In [Naji and Brown, 2007], the authors study lateral diffusion on a Helfrich membrane undergoing thermal fluctuations. They identify two limiting regimes: the diffusive limit (homogenization) of a diffusion on a quenched surface and the annealed limit (averaging) of diffusion on a rapidly fluctuating membrane, based on a formal analysis of the Fokker-Planck equation describing the evolution of the system, using the methodology of [Risken, 1996]. They then use numerical methods to study the dynamics of the intermediate regimes where there is no separation of scales. However, to our knowledge, there are no studies which adopt a rigorous multiscale approach to solving this problem, nor are we aware of any work which unifies the study of lateral diffusion on surfaces with both rapid spatial and temporal fluctuations in a single framework. Moreover, we are not aware of any study which makes use of multiscale methods to compute the effective diffusion coefficient directly rather than using direct numerical simulation of the multiscale process, with the exception of [Abdulle and Schwab, 2005] in which the authors describe an HMM (heterogenous multiscale method) scheme for computing the solution of an elliptic partial differential equation (PDE) on a static surface possessing fine locally-periodic undulations and rigorously prove convergence of the scheme.

1.1 OVERVIEW OF THE THESIS

We briefly describe the organisation of this thesis and summarise the contents of each chapter.

Chapter 2 provides an overview of the basic results regarding lateral diffusion on quasi-planar surfaces, both from a stochastic differential equation (SDE) perspective and a PDE perspective. We also describe a widely-used two-dimensional continuum model for modelling a fluctuating membrane, based on the Canham-Helfrich elastic free energy [Canham, 1970; Helfrich et al., 1973]. After non-dimensionalisation of the coupled equation describing the particle-membrane evolution, we identify a natural scaling of the problem, and show that for some choices of parameters, the system is well described by its annealed disorder limit (see Naji and Brown [2007]; Reister and Seifert [2007]). We also introduce three simple models which will be studied in the subsequent chapters.

In Chapter 3 we consider the simplest model for surface fluctuations, namely where the surface is described by a static, periodic function. We identify a natural scaling for the surface in terms of a small-scale parameter ϵ. Using classical homogenisation techniques, we derive a limiting equation for a diffusion on this surface in the limit of vanishing ϵ, and use various standard results from classical homogenisation theory to obtain expressions or approximations for D.

In Chapter 4 we consider the second model where the surface is described by a time-independent, stationary random field which is ergodic with respect to spatial translations. We consider the problem of diffusion on a high-frequency, low-amplitude rescaling of this surface. Using results from stochastic homogenisation theory we are able to generalise all the results from the previous chapter to this case.

In Chapter 5 we consider a simple model for a time-dependent, spatially fluctuating surface possessing rapid spatial and temporal fluctuations, which is a generalisation of the fluctuating Helfrich elastic membrane model. The limiting behaviour will depend on the relative speed between the spatial and temporal fluctuations, which is determined by two parameters α and β, respectively. We consider four natural choices of α and β, and for each case study the effective properties of the corresponding limit processes. In this chapter we only provide formal justifications of the homogenisation results, using formal perturbation expansions. So as not to break the flow of this chapter the rigorous justification of these results will be deferred to Appendix A.

In Chapter 6, for the particular case of the thermally excited Helfrich elastic membrane model, we consider the remaining possible choices of α and β and enumerate all the possible distinguished limits of this model.

In Appendix A we prove the homogenisation results for the four scaling limits described in Chapter 5 using probabilistic methods.

In Appendix B we describe some existing results which are used in the proofs of this thesis.

Chapter 2

BACKGROUND

2.1 LATERAL DIFFUSION ON FLUCTUATING SURFACES

In this section we describe the formulation of Brownian motion moving on a time-dependent surface embedded in $\mathbb{R}^{d+1}$. We are primarily interested in quasi-planar membranes, so we will restrict our attention to surfaces which can be represented in the *Monge parametrisation*, that is, surfaces which can be expressed as the graph of a sufficiently smooth function $H : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$. Such a surface $S(t)$ can then be parametrised over $\mathbb{R}^d$ by $J : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^{d+1}$ given by

$$J(x, t) = (x, H(x, t)).$$

The function H is known as the *Monge gauge*. Although this choice of parametrisation restricts the representable surfaces (overhangs, in particular, are prohibited), it greatly simplifies the exposition that follows. In local coordinates $x \in \mathbb{R}^d$, the metric tensor of $S(t)$ induced from $\mathbb{R}^{d+1}$, can be written as

$$G(x, t) = I + \nabla H(x, t) \otimes \nabla H(x, t) \tag{2.1}$$

and the infinitesimal surface area element is given by $\sqrt{|G|(x, t)}$, where

$$|G|(x, t) := \det\big(G(x, t)\big) = 1 + |\nabla H(x, t)|^2. \tag{2.2}$$

It is clear that for any unit vector $e \in \mathbb{R}^d$,

$$1 \leq e \cdot G(x, t)e \leq |G|(x, t), \quad \text{for all } x \in \mathbb{R}^d,$$

so that G^{-1} is symmetric, positive definite (though not necessarily uniformly so). Given $F : \mathbb{R}^{d+1} \to \mathbb{R}$ smooth in a neighbourhood of $S(t)$, the tangential gradient of F is given in local coordinates by

$$\nabla_{S(t)} F(J(x, t)) = \mathcal{P}(x, t) \nabla F(J(x, t)) = \nabla J(x, t)^\top G^{-1}(x, t) \nabla (F \circ J)(x, t).$$

Here, $\mathcal{P}(x,t)$ projects vectors in $\mathbb{R}^{d+1}$ onto the tangent space of $S(t)$ at local coordinate x, that is,

$$\mathcal{P}(x,t) = I - \nu(x,t) \otimes \nu(x,t),$$

where $\nu(x,t)$ is the surface unit normal of $S(t)$. The tangential divergence $\nabla_{S(t)}\cdot$ is then obtained from the tangential gradient by contraction. The generalisation of the Laplace operator to curved surfaces is the Laplace-Beltrami operator $\Delta_{S(t)}$ which is given by

$$\Delta_{S(t)} F = \nabla_{S(t)} \cdot \nabla_{S(t)} F.$$

One can show [Deckelnick et al., 2005; Dziuk and Elliott, 2013] that in local coordinates, $\Delta_{S(t)}$ acts on smooth functions $F \in C^2(\mathbb{R}^{d+1})$ as follows

$$\Delta_{S(t)} F(J(x,t)) = \frac{1}{\sqrt{|G|\,(x,t)}} \nabla \cdot \left(\sqrt{|G|\,(x,t)} G^{-1}(x,t) \nabla \left(F \circ J\right)(x,t) \right),$$

for $x \in \mathbb{R}^d$. We thus define the operator $\mathcal{L}_t$ acting on functions $f \in C^2(\mathbb{R}^d)$ to be the local coordinate representation of the Laplace Beltrami operator:

$$\mathcal{L}_t f(x,t) = \frac{1}{\sqrt{|G|\,(x,t)}} \nabla \cdot \left(\sqrt{|G|\,(x,t)} G^{-1}(x,t) \nabla f(x,t) \right), \quad \text{for } x \in \mathbb{R}^d. \tag{2.3}$$

It is clear that $\Delta_{S(t)} F(J(x,t)) = \mathcal{L}_t \left(F \circ J\right)(x,t)$, for all $x \in \mathbb{R}^d$, $t \geq 0$ and for all $F \in C^2(\mathbb{R}^{d+1})$. Notice that for a flat surface, for which $H \equiv 0$, the operator reduces to the standard Laplace operator on $\mathbb{R}^d$.

2.1.1 BROWNIAN MOTION ON AN EVOLVING SURFACE

While the properties of Brownian motion on static surfaces have been widely studied in the applied literature (see [Van Den Berg and Lewis, 1985; Sbalzarini et al., 2006; Almeida and Vaz, 1995; Naji and Brown, 2007]), Brownian motion on time-dependent surfaces has been given less consideration. In [Naji and Brown, 2007] the authors formally derive the over-damped Langevin equation for diffusion on a surface in the Monge gauge as the limit of a random walk constrained to the surface. In [Coulibaly-Pasquier, 2011], the author provides a rigorous definition of Brownian motion on a manifold with a time-dependent metric. As we are working entirely in the Monge gauge we provide the following natural definition of Brownian motion on a fluctuating Monge-gauge surface, which is equivalent to that given in [Coulibaly-Pasquier, 2011] in the Monge-gauge representation.

Definition 2.1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $S(t)$ be a time-dependent surface, with corresponding Monge gauge $H(x,t)$, where $H(\cdot,t) \in C^2(\mathbb{R}^d)$. Then, an $\mathbb{R}^d$-valued process $X_x(t)$ defined on $\Omega \times [0,T)$ is called a Brownian motion on $S(t)$ started at $X_x(0) = x \in \mathbb{R}^d$, if $X(t)$ is continuous, adapted, and if for every smooth function $f : \mathbb{R}^d \to \mathbb{R}$,*

$$f(X_x(t)) - f(x) - \int_0^t \mathcal{L}_s f(X_x(s))\, ds,$$

is a local martingale (see Definition 5.5, [Karatzas and Shreve, 1991]), where $\mathcal{L}_s$ is the Laplace-Beltrami operator (2.3) in local coordinates on $\mathbb{R}^d$.

We note that in the case where $H \equiv 0$, Definition 2.1.1 reduces to standard Brownian motion on $\mathbb{R}^d$.

Let $S(t)$ be a time-dependent surface with Monge gauge $H(x,t)$ such that for $t \geq 0$, $H(\cdot,t) \in C^2(\mathbb{R}^d)$. Define $X_x(t)$ to be the solution of the following Itô SDE

$$\begin{aligned} dX_x(t) &= F(X_x(t),t)\,dt + \sqrt{2\Sigma(X_x(t),t)}\,dB(t), \\ X_x(0) &= x, \end{aligned} \tag{2.4}$$

where

$$F(x,t) = \frac{1}{\sqrt{|G(x,t)|}} \nabla_x \cdot \left(\sqrt{|G(x,t)|} G^{-1}(x,t) \right), \tag{2.5}$$

and

$$\Sigma(x,t) = G^{-1}(x,t), \tag{2.6}$$

and $B(\cdot)$ is a standard $\mathbb{R}^d$-valued Brownian motion. Here $\sqrt{\Sigma(x,t)}$ denotes the unique positive-definite square root of $\Sigma(x,t)$. The existence of a unique strong solution (in the sense of Section 5.2 of [Karatzas and Shreve, 1991]) depends on the form of $H(x,t)$, and we will verify that independently for each surface fluctuation model we consider. For now we assume that there exists a unique strong solution of the SDE (2.4). Then by Itô's formula (Theorem 3.3, [Karatzas and Shreve, 1991]), for smooth $f : \mathbb{R}^d \to \mathbb{R}$:

$$\begin{aligned} f(X_x(t)) - f(x) = \\ &\int_0^t \frac{1}{\sqrt{|G|\,(X_x(s),s)}} \nabla_x \cdot \left(\sqrt{|G|\,(X_x(s),s)}\, G^{-1}(X_x(s),s) \right) \nabla_x f(X_x(s))\,ds \\ &+ \int_0^t G^{-1}(X_x(s),s) : \nabla_x \nabla_x f(X_x(s))\,ds \\ &+ \int_0^t \sqrt{2\,G^{-1}(X_x(s),s)}\,\nabla_x f(X_x(s))\,dB(s) \\ &= \int_0^t \mathcal{L}_s f(X_x(s))\,ds + M(t), \end{aligned}$$

where $M(t)$ is a local martingale. It follows that $X_x(t)$ satisfies the conditions of Definition 2.1.1 to be a Brownian motion on the evolving Monge-gauge surface $S(t)$.

Independently, we may derive from first principles the evolution equation for the probability density $\rho(z,t)$ of a particle undergoing Brownian motion on a time-dependent surface given expressed in the Monge gauge. From this, we can then recover the same SDE (2.4).

To this end, consider a particle undergoing Brownian motion moving on the time-dependent surface $S(t)$, and suppose that the process possesses a density $\rho(t,z)$ for $z \in S(t)$. Let Θ be an arbitrary bounded region in $\mathbb{R}^d$ with smooth boundary, and let $\mathcal{M}(t)$ be the corresponding region on the fluctuating surface, that is,

$$\mathcal{M}(t) = J(\Theta, t).$$

The density $\rho(z,t)$ is conserved on the surface $S(t)$ for all t such that

$$\int_{S(t)} \rho(z,t)\, dz = 1, \qquad t \geq 0.$$

Moreover, we expect that ρ flows from regions of low concentration on $S(t)$ to regions of high concentration, that is we expect that the density flows with local Fickian flux $-D_0 \nabla_{S(t)} \rho(z,t)$ where $\nabla_{S(t)}$ is the tangential derivative on the surface $S(t)$, and where D_0 is a scalar diffusivity constant. It follows that $\rho(z,t)$ satisfies the following equation

$$\frac{\partial}{\partial t} \int_{\mathcal{M}(t)} \rho(z,t)\, dz = D_0 \int_{\partial\mathcal{M}(t)} \nabla_{S(t)} \rho(z,t) \cdot n(z,t)\, dz = \int_{\mathcal{M}(t)} D_0 \Delta_{S(t)} \rho(z,t)\, dz,$$

where $n(z,t)$ is the conormal exterior vector along the boundary of $\mathcal{M}(t)$. See [Deckelnick et al., 2005] for details. Changing variables from z to local coordinates x induces a change of measure $dz = \sqrt{|G|\,(x,t)}dx$ where $|G|$ is given by (2.2). We can thus rewrite the above equation in local coordinates as

$$\begin{aligned} &\frac{\partial}{\partial t} \int_{\Theta} \rho(J(x,t),t) \sqrt{|G|\,(x,t)}\, dx \\ &\quad = \int_{\Theta} D_0 \nabla_x \cdot \left(\sqrt{|G|\,(x,t)} G^{-1}(x,t) \nabla_x \left(\rho \circ J\right)(x,t) \right)\, dx. \end{aligned}$$

As we are only interested in the trajectory of the diffusion process projected onto the plane, we weight the density ρ with the surface area element $\sqrt{|G|\,(x,t)}$ to compensate for the local changes in area of the surface. To this end, define the density $q : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ by

$$q(x,t) := \rho\left(J\left(x,t\right),t\right) \sqrt{|G|\,(x,t)}.$$

It is straightforward to check that $\int_{\mathbb{R}^d} q(x,t)\, dx = 1$ for all time t. Substituting $q(x,t)$ in the previous equation, and noting that Θ is arbitrary, we obtain the following pointwise relationship for q on $\mathbb{R}^d$:

$$\frac{\partial}{\partial t} q(x,t) = D_0 \nabla_x \cdot \left(\sqrt{|G|\,(x,t)} G^{-1}(x,t) \nabla_x \left(\frac{q(x,t)}{\sqrt{|G|\,(x,t)}} \right) \right) = D_0 \mathcal{L}_t^* q(x,t), \quad (2.7)$$

where $\mathcal{L}_t^*$ denotes the adjoint of the operator defined in (2.3). We note that equation (2.7) is the forward Kolmogorov (Fokker-Planck) equation for a diffusion process with infinitesimal generator given by $\mathcal{L}_t$, [Friedman, 2006, Chapter 6].

2.2 LATERAL DIFFUSION ON A THERMALLY FLUCTUATING HELFRICH MEMBRANE

In this section we introduce a particular surface fluctuation model which describes at the mesoscopic level the shape of a quasi-planar bilayer membrane undergoing thermal fluctuations in a low Reynolds number hydrodynamic medium, as derived by [Granek, 1997].

Modelling the bilayer membrane as a two dimensional graph over $[0, L]^2$ with periodic boundary conditions, we assume that the equilibrium configuration of the membrane is described by the bending energy proposed in [Helfrich et al., 1973] and [Canham, 1970]:

$$\mathcal{E}_{\text{bending}}[H] = \frac{1}{2}\int_{[0,L]^2} \Big(\kappa\mathcal{K}(x)^2 + \overline{\kappa}\mathcal{K}_G(x)\Big)\sqrt{|G|\,(x)}\,dx,$$

where $G(x)$ is the metric tensor of the surface with Monge gauge $H(x)$ in local coordinates. Here $\mathcal{K}(x)$ and $\mathcal{K}_G(x)$ are the mean curvature and Gaussian curvature of S respectively, [Deckelnick et al., 2005]. The two moduli κ and $\overline{\kappa}$ are the bending modulus and saddle-splay modulus respectively. Since we assume that the surface does not undergo topological changes during its evolution, by the Gauss-Bonnet theorem, the integral of the Gaussian curvature depends only on the boundary values, and so can be neglected from the energy functional.

The Helfrich Hamiltonian is commonly defined as elastic bending energy plus an additional term to penalise excess surface area

$$\mathcal{H}[H] = \int_{[0,L]^2} \left(\frac{\kappa}{2}\mathcal{K}^2(x) + \frac{\sigma}{2}\right)\sqrt{|G|\,(x)}\,dx, \tag{2.8}$$

where σ is the surface tension.

For small deformations, $|\nabla H(x)| \ll 1$, we can make the following approximation for the mean curvature:

$$\mathcal{K}(x) = \nabla\cdot\left(\frac{\nabla H(x)}{\sqrt{|G|\,(x)}}\right) \approx \Delta H(x), \qquad \text{for } x \in [0, L]^2$$

and for the local surface element:

$$\sqrt{|G(x)|} \approx 1 + \frac{1}{2}\left|\nabla H(z)\right|^2, \qquad \text{for } x \in [0, L]^2$$

so that the Helfrich free energy can be approximated by

$$\mathcal{H}[H] = \frac{1}{2}\int_{[0,L]^2} \kappa\left(\Delta H(x)\right)^2 + \sigma\left|\nabla H(x)\right|^2\,dx + \frac{\sigma L^2}{2}. \tag{2.9}$$

The constant term is discarded, leaving the form of the Helfrich free energy which will be used throughout this thesis.

Using linear response theory [Van and Carolyn, 2008] we can describe the dynamics of the thermally excited membrane close to equilibrium by

$$\frac{dH(t)}{dt} = RAH(t) + \zeta(t), \tag{2.10}$$

where

$$AH(t) = -\frac{\delta \mathcal{H}}{\delta H}[H(t)] = -\kappa \Delta^2 H(t) + \sigma \Delta H(t),$$

and $\zeta(t)$, a Gaussian random field white in time and with spatial fluctuations having mean zero and covariance operator $2\ (k_B T)\ R$, where k_B is the Boltzmann constant, and T is the temperature. This construction ensures that the dynamics in (2.10) satisfies the fluctuation-dissipation relation required to ensure that, formally, the invariant measure is proportional to $\exp(-\mathcal{H}/(k_B T))$.

The operator R controls the characteristic time for Fourier modes of the membrane to return to equilibrium, and will encode the non-local interactions of the membrane through the hydrodynamic medium. Using an approach analogous to [Doi and Edwards, 1988] for polymer dynamics, R is approximated by

$$Rf(x) = (\Lambda * f)(x), \qquad f \in L^2_{per}([0, L]^2)$$

where $*$ denotes convolution, and $\Lambda(x)$ is given by the diagonal part of the Oseen tensor:

$$\Lambda(x) := \frac{1}{8\pi\lambda\,|x|},$$

where λ is the viscosity of the surrounding medium.

As we are only interested in the local displacement of the surface about the plane, we shall assume that surface configurations have mean 0. To this end, denote by $L^2_{per}([0, L]^2; \mathbb{R})$ the space of all square-integrable functions periodic on $[0, L]^2$ with mean 0 and let $\{e_{L,k} \mid k \in \mathbb{K}_\infty\}$ be the standard Fourier basis for $L^2_{per}([0, L]^2; \mathbb{R})$, indexed by

$$\mathbb{K}_\infty := \mathbb{Z}^2 \setminus \{(0, 0)\}.$$

It is straightforward to check that the invariant measure of $H(t)$ is given by $\mathcal{N}(0, \mathcal{C})$ where

$$\mathcal{C} \propto k_B T \sum_{k \in \mathbb{K}_\infty} \left(\kappa\, |2\pi k|^4 + \sigma L^2\, |2\pi k|^2\right)^{-1} e_{L,k}(x) \otimes e_{L,k}(x).$$

The operator $\mathcal{C}^{-\frac{1}{2}}$ satisfies Assumptions 2.9 (i)-(iv) of [Stuart, 2010], so that its spectra grows commensurately with those of $-\Delta$. It follows from Lemma 6.25 of [Stuart, 2010] that the stationary realisations of the random field will be Hölder continuous with exponent $\alpha < 1$, but not for $\alpha = 1$. This implies that realisations are not sufficiently regular to allow well-defined tangents at every point on the surface. Indeed, $H(x,t)$ will be almost surely nowhere differentiable so that it is not possible to consider a laterally diffusive process on a realisation of this random field. To ensure that the realisations of the surface are sufficiently smooth to permit lateral diffusion, we must either add a further regularisation term to the Helfrich Hamiltonian, or introduce an ultraviolet cut-off. Having an ultraviolet cut-off reduces the SPDE to a finite dimensional system. Since all the previous work (e.g. [Granek, 1997; Lin and Brown, 2004; Reister and Seifert, 2007; Naji and Brown, 2007]) make an ultraviolet cut-off, we adopt the same approach. To this end, we set $\langle e_{L,k}, \mathcal{C} e_{L,k}\rangle = 0$ for wave numbers $k \notin \mathbb{K}$, where

$$\mathbb{K} = \{k \in \mathbb{K}_\infty \mid |k| \leq c\},$$

for some fixed constant $c > 0$ and define $K = |\mathbb{K}|$.

Substituting $H(x,t)$ of the form

$$H(x,t) = \frac{1}{L^2} \sum_{k\in\mathbb{K}} \eta_k(t) e_{L,k}(x), \tag{2.11}$$

into (2.10) we note that the SPDE diagonalises, and we obtain the following system of SDEs for the Fourier modes $\eta_k(t)$:

$$d\eta_k(t) = -\frac{\kappa\,|2\pi k/L|^3 + \sigma\,|2\pi k/L|}{4\lambda}\eta_k(t)\,dt + \sqrt{\frac{k_B T L^3}{2\lambda\,|2\pi k|}}dW_k(t), \tag{2.12}$$

where $W_k(t) = \frac{1}{\sqrt{2}}\left(W_k^r(t) + i\,W_k^i(t)\right)$ for standard real valued independent Brownian motions W_k^r and W_k^i. For $k \neq k'$, $W_k(t)$ and $W_{k'}(t)$ are independent except for the *reality constraint* that

$$W_{-k}(t) = W_k^*(t),$$

which guarantees that $H(x,t)$ is real-valued for all time. It is straightforward to see that each Fourier mode $\eta_k(t)$ has invariant measure

$$\mu_k = \mathcal{N}\left(0, \frac{2\,k_B T L^6}{\kappa\,|2\pi k|^4 + \sigma L^2\,|2\pi k|^2}\right),$$

and it is reasonable to assume that $\eta_k(0) \sim \mu_k$.

Consider a particle with macroscopic diffusion coefficient D_0, undergoing Brownian motion on the time-dependent surface $S(t)$ with Monge-Gauge $H(x,t)$. Following Section 2.1.1, the evolution of the particle trajectory is described by the following SDE

$$\begin{aligned} dX(t) = {} & \frac{D_0}{\sqrt{|G|\,(X,t)}}\nabla\cdot\left(\sqrt{|G|\,(X(t),t)}G^{-1}(X(t),t)\right)\,dt \\ & + \sqrt{2D_0 G^{-1}(X(t),t)}\,dB(t), \end{aligned} \tag{2.13}$$

where $G(x,t)$ is the metric tensor corresponding to the Monge Gauge $H(x,t)$, i.e.

$$G(x,t) = I + \nabla H(x,t) \otimes \nabla H(x,t),$$

for $H(x,t)$ given by (2.11).

Let $\left\{ e_k(x) = e^{2\pi i k \cdot x} \,\middle|\, k \in \mathbb{K}_\infty \right\}$ be the standard periodic fourier basis functions for $L^2_{per}([0,1]^2)$. Setting

$$x = Lx^*, \quad t = Tt^*, \quad \text{and} \quad X(Tt^*) = LX^*(t^*),$$

and choosing $T = L^2/D_0$, we can non-dimensionalise (2.13), noting the units of the parameters in Table 2.1 we obtain the following non-dimensional system of SDEs:

$$\begin{aligned} dX^*(t^*) &= \frac{1}{\sqrt{|G^*|\,(X(t^*),t^*)}} \nabla \cdot \left(\sqrt{|G^*|\,(X^*(t^*),t^*)} G^{*-1}(X^*(t^*),t^*) \right) dt^* \\ &\quad + \sqrt{2G^{*-1}(X^*(t^*),t^*)}\, dB(t^*), \\ d\eta^*(t^*) &= -\frac{1}{\epsilon}\Gamma \eta^*(t^*)\, dt + \sqrt{\frac{2\Gamma\Pi}{\epsilon}} dW(t^*), \end{aligned} \tag{2.14}$$

where

$$\Gamma = \operatorname{diag}\left(\frac{\kappa^* |2\pi k|^4 + \sigma^* |2\pi k|^2}{|2\pi k|} \right)_{k \in \mathbb{K}}, \tag{2.15}$$

$$\Pi = \operatorname{diag}\left(\frac{1}{\kappa^* |2\pi k|^4 + \sigma^* |2\pi k|^2} \right)_{k \in \mathbb{K}}, \tag{2.16}$$

and where κ^*, σ^* and ϵ are dimensionless constants given by

$$\kappa^* = \frac{\kappa}{k_B T}, \qquad \sigma^* = \frac{\sigma L^2}{k_B T},$$

and

$$\epsilon = \frac{4\lambda L D_0}{k_B T}.$$

The matrix $G(x,t)$ is the metric tensor of the surface with Monge gauge

$$H^*(x,t) = \sum_{k \in \mathbb{K}} \eta^*_k e_k(x).$$

Quantity	Symbol	Units
Temperature	$k_B T$	J
System size	L	m
Environment viscosity	λ	$k_B T s m^{-3}$
Bending stress	κ	Nm
Surface tension	σ	Nm^{-1}
Molecular Diffusivity	D_0	$m^2 s^{-1}$

Table 2.1: Model parameters and units

The parameter $\chi = \epsilon^{-1}$ had already been considered in [Naji and Brown, 2007] where it was called the *dynamic coupling parameter* because it controls the scale separation between the diffusion and the surface fluctuations. As discussed in [Naji and Brown, 2007], for the particular case of band-3 protein diffusion on a human red blood cell, the typical values of parameters result in $\epsilon \approx 0.3$, which suggests that ϵ is an appropriate small-scale parameter. Of course, the value of ϵ will vary greatly for different scenarios.

The limiting behaviour as $\epsilon \to 0$, for the scaling given in equation (2.14) will be discussed in detail in Section 5.1.2. However, it is a sufficiently simple model to study the limiting behaviour in other scalings, and so we revisit this model in Chapter 5 and more exhaustively in Chapter 6. The main question which we will attempt to answer for each scaling is the dependence of the macroscopic diffusion coefficient on the system parameters κ^*, σ^* and the ultraviolet cutoff c (or equivalently K).

2.3 MODELS OF RAPIDLY FLUCTUATING SURFACES

In this section we introduce the three main models of fluctuating Monge-gauge surfaces which will be studied in the thesis. For each model we will derive a system of coupled stochastic differential equations which describe the joint evolution of the particle and the surface. Additionally, we obtain the corresponding Kolmogorov equations for the stochastic system. Although we adopt a primarily probabilistic approach to homogenisation, the PDE formulation is convenient to identify the homogenised equations using formal perturbation expansions. We make use of the PDE approach in Chapters 3 and 5, deferring the rigorous probabilistic proofs to Appendix A.

2.3.1 DIFFUSION ON A STATIC, SPATIALLY PERIODIC SURFACE

The first model we consider provides the simplest possible description of a rough interface. We represent the interface as a time-independent surface consisting of a low amplitude, high-frequency, periodic perturbation of the plane. More specifically, we consider a surface S^ϵ with Monge gauge given by

$$h^\epsilon(x) = \epsilon h\left(\frac{x}{\epsilon}\right), \tag{2.17}$$

where ϵ is a small scale parameter and h is a sufficiently smooth real-valued function on $\mathbb{R}^d$ such that h and its derivatives are periodic with period 1 in every direction.

We note that as $\epsilon \to 0$, the average surface area is conserved, in the sense that it remains $O(1)$ with respect to ϵ, which suggests that (2.17) is the natural scaling for this problem. This is illustrated for the 1D case in Figure 2.1, which plots the surface S^ϵ given by $h^\epsilon(x)$ for $h(x) = \sin(2\pi x)$ and $\epsilon = 1, 0.1$ and 0.05. Let Z denote the arc length of the surface over $[-1, 1]$, and consider the projected trajectory $X^\epsilon(t)$ of a particle undergoing lateral diffusion on S^ϵ starting from 0. The escape time of $X^\epsilon(t)$ from $[-1, 1]$ is equal to the expected escape time of a free $\mathbb{R}$-valued Brownian motion from the interval $[-Z, Z]$, which is $\frac{Z^2}{2}$. Taking $\epsilon = \frac{1}{n} \to 0$, the expected escape time of X^ϵ remains $\frac{Z^2}{2}$ in the limit, which implies that in the limit, the law of X^ϵ behaves identically to a free Brownian motion on $\mathbb{R}$ with constant diffusion coefficient $\frac{1}{Z^2}$. We note that any other scaling would result in the surface area going to 0 or ∞ as $\epsilon \to 0$, which suggests that (2.17) is the only good scaling for this problem.

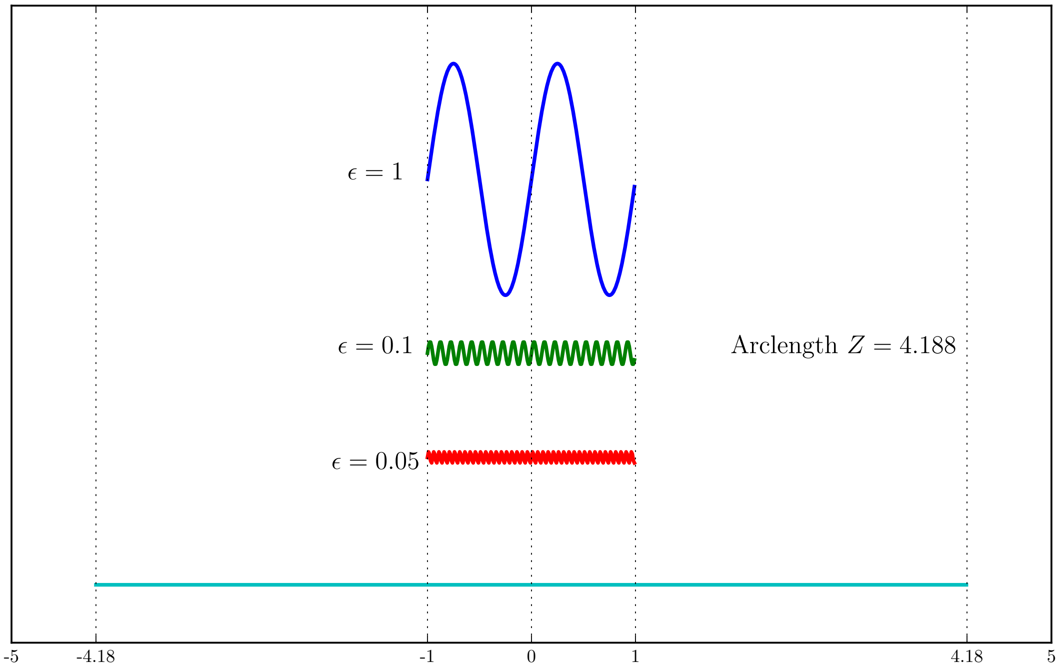


Figure 2.1: The curve S^ϵ with Monge-Gauge $h^\epsilon(x) = \epsilon \sin(2\pi x/\epsilon)$. The three curves in this figure have the same arc-length $Z \approx 4.19$.

It is straightforward to see that S^ϵ has metric tensor $g^\epsilon(x) = g(x/\epsilon)$, where

$$g(x) = I + \nabla h(x) \otimes \nabla h(x), \qquad x \in \mathbb{T}^d. \tag{2.18}$$

Consider a particle diffusing along the surface S^ϵ and let $X^\epsilon(t)$ denote the projection onto the plane. Following the derivation in Section 2.1.1 with $H(x,t) = h^\epsilon(x)$, the evolution of $X^\epsilon(t)$ is given by the following Itô SDE

$$dX^\epsilon(t) = \frac{1}{\epsilon} F(X^\epsilon(t)/\epsilon)\, dt + \sqrt{2\Sigma(X^\epsilon(t)/\epsilon)}\, dB(t), \tag{2.19}$$

where $F(x)$ and $\Sigma(x)$ are as in (2.5) and (2.6), but with the dependence on t suppressed, i.e.

$$F(x) = \frac{1}{\sqrt{|g|\,(x)}} \nabla \cdot \left(\sqrt{|g|\,(x)} g^{-1}(x) \right),$$

and

$$\Sigma(x) = g^{-1}(x).$$

Equivalently, one can consider an observable

$$u^\epsilon(x,t) = \mathbb{E}\left[u(X^\epsilon(t) \,|X^\epsilon(0) = x \right],$$

where $X^\epsilon(t)$ is a Brownian motion on a surface S^ϵ given by $h^\epsilon(x)$ and where $u \in C_b(\mathbb{R}^d)$. The observable $u^\epsilon(x,t)$ evolves according to the backward Kolmogorov equation [Friedman, 2006, Chapter 6]:

$$\begin{aligned} \frac{\partial u^\epsilon(x,t)}{\partial t} &= \mathcal{L}^\epsilon u^\epsilon(x,t), \qquad && (x,t) \in \mathbb{R}^d \times (0,T], \\ u^\epsilon(x,t) &= u(x), && (x,t) \in \mathbb{R}^d \times \{0\}. \end{aligned} \tag{2.20}$$

where

$$\mathcal{L}^\epsilon f(x) = \frac{1}{\sqrt{|g|\,(x/\epsilon)}} \nabla_x \cdot \left(\sqrt{|g|\,(x/\epsilon)}) g^{-1}(x/\epsilon) \nabla_x f(x) \right). \tag{2.21}$$

In Chapter 3 we study the asymptotic behaviour of (2.19) and (2.20) in the limit as $\epsilon \to 0$. In particular we study how properties of the surface influence the limiting diffusion coefficient. While the results derived in the chapter are straightforward, they form a basis for the more involved results derived in subsequent parts of this thesis. Similar models have been studied previously in the literature, in particular [Aizenbud and Gershon, 1985], [Halle and Gustafsson, 1997] and [Naji and Brown, 2007], all of which were in the context of modelling biological membranes. The simplicity of this model is due to the fact that it is amenable to classical periodic homogenization methods (such as [Bensoussan et al., 1978; Jikov et al., 1994; Pavliotis and Stuart, 2008]), which allow us to very easily obtain variational bounds on the effective diffusion coefficient in terms of the Monge gauge h.

2.3.2 DIFFUSION ON A STATIC SURFACE GENERATED BY A SPATIALLY ERGODIC RANDOM FIELD

We can generalise the static, periodic fluctuation model of the previous section considerably by replacing the periodic function $h(x)$ in (2.19) with a realisation of a random field $h : \mathbb{R}^d \to \mathbb{R}$ with probability measure $\mathbb{P}$. We assume that the random field has mean 0, and is stationary, that is,

$$\mathbb{E}_{\mathbb{P}}\left[h(x)h(y)\right] = C(x-y), \qquad x, y \in \mathbb{R}^d,$$

for some non-negative function $C : \mathbb{R}^d \to \mathbb{R}$. Moreover, we assume that the random field is ergodic with respect to spatial translations, that is, expectations with respect to $\mathbb{P}$ can be replaced by spatial averages (this will be made rigorous in Chapter 4). We also assume that realisations of $h(x)$ are $\mathbb{P}$-almost surely bounded and with bounded derivatives up to order 3.

As in the periodic case, for a fixed realisation h of the random field we consider the surface S^ϵ with Monge gauge $h^\epsilon(x) = \epsilon h(x/\epsilon)$, where $\epsilon > 0$ is a small scale parameter. The argument as to why this is the natural scaling for the problem is identical to that in the periodic case.

For a given realisation of h, if we denote by $X^\epsilon(t)$ the projected trajectory of a particle diffusing laterally on the surface S^ϵ, then the evolution of $X^\epsilon(t)$ will also be described by the Itô SDE (2.19) and the corresponding backward Kolmogorov equation for an observable $u^\epsilon(x,t)$ is given by (2.20).

Formulated in this way, the problem is amenable to standard stochastic homogenisation methods, such as those in [Papanicolaou et al., 1979; Kipnis and Varadhan, 1986; Komorowski et al., 2012]. In Chapter 4, we use the stochastic homogenisation approach to study the behaviour of $X^\epsilon(t)$ and $u^\epsilon(x,t)$ as $\epsilon \to 0$. Given the assumptions above we can show that in the limit as $\epsilon \to 0$, $X^\epsilon(t)$ converges to a free Brownian motion on $\mathbb{R}^d$ with a constant diffusion coefficient which is determined by the statistics of the random field (but independent of any particular realisation).

2.3.3 DIFFUSION ON A TIME-DEPENDENT RANDOM SURFACE

The third model we consider is a model for a time-dependent surface possessing both rapid spatial and temporal fluctuations, based on the model discussed in Section 2.2 for a thermally fluctuating Helfrich membrane. We consider surfaces whose Monge gauge is a time-dependent random field $H(x,t)$ such that the following assumptions hold:

1. For each $t \geq 0$, $H(x,t)$ is smooth in x;
2. For each $t \geq 0$, $H(x,t)$ and its derivatives are periodic in x with period L_H;
3. For each $x \in \mathbb{R}^d$, $H(x,t)$ has characteristic time T_H.

Consider a particle diffusing on a realisation of the surface $H(x,t)$ with an isotropic molecular diffusion coefficient D_0. Let $X(t)$ denote the projected trajectory on $\mathbb{R}^d$, and let L and T be the system length and time scales at which the process $X(t)$ is being observed. We introduce the notation

$$
\begin{aligned}
x = Lx^*, \qquad t = Tt^*,\\
X(Tt^*) = LX^*(t^*),\\
H(x,t) = \tilde{H}H^*\left(\frac{x}{L_H}, \frac{t}{T_H}\right),
\end{aligned}
$$

where $\tilde{H}$ is a scaling constant, so that rescaled function H^* has period 1 in space. Define the parameters δ and τ to be

$$
\delta = \frac{L_H}{L} \quad \text{and} \quad \tau = \frac{T_H}{T}, \tag{2.22}
$$

which quantify the scale separation between the diffusion process $X(t)$ and the spatial and temporal fluctuations respectively.
Substituting these definitions into (2.4) we obtain the following SDE

$$
\begin{aligned}
&dX^*(t^*)\\
&= \frac{1}{\delta}\frac{D_0 T}{L^2}\frac{1}{\sqrt{|G^*|\left(\frac{X^*(t^*)}{\delta}, \frac{t^*}{\tau}\right)}}\nabla_y \cdot \left(\sqrt{|G^*|\left(\frac{X^*(t^*)}{\delta}, \frac{t^*}{\tau}\right)}G^{*-1}\left(\frac{X^*(t^*)}{\delta}, \frac{t^*}{\tau}\right)\right) dt^*\\
&\quad + \sqrt{\frac{2D_0 T}{L^2}G^{*-1}\left(\frac{X^*(t^*)}{\delta}, \frac{t^*}{\tau}\right)}dB(t^*),
\end{aligned} \tag{2.23}
$$

where

$$
G^*(y,s)_{ij} = \delta_{ij} + \left(\frac{\tilde{H}}{L_H}\nabla_y H^*(y,s)\right) \otimes \left(\frac{\tilde{H}}{L_H}\nabla_y H^*(y,s)\right). \tag{2.24}
$$

By choosing $T = \frac{L^2}{D_0}$ and $\tilde{H} = L_H$, we obtain a dimensionless SDE where parameters δ and τ measure the spatial and temporal scale separation respectively. To reflect the assumption of rapid spatial and temporal fluctuations we assume that $\delta \ll 1$ and $\tau \ll 1$.

Taking the limit of δ and τ going to zero will give different limits depending on the relationship between these two parameters. To this end, we assume that both τ and δ depend on a common parameter ϵ as follows

$$
\delta = \epsilon^\alpha \qquad \text{and} \qquad \tau = \epsilon^\beta, \tag{2.25}
$$

for scaling parameters $\alpha > 0$ and $\beta \in \mathbb{R}$. By assuming (2.25), dropping all the stars and making the dependence of X on ϵ explicit, equation (2.23) can be written as follows

$$\begin{aligned} dX^\epsilon(t) = &\frac{1}{\epsilon^\alpha}\frac{1}{\sqrt{|G|\left(\frac{X^\epsilon(t)}{\epsilon^\alpha},\frac{t}{\epsilon^\beta}\right)}}\nabla_y\cdot\left(\sqrt{|G|}G^{-1}\right)\left(\frac{X^\epsilon(t)}{\epsilon^\alpha},\frac{t}{\epsilon^\beta}\right)dt \\ &+\sqrt{2G^{-1}\left(\frac{X^\epsilon(t)}{\epsilon^\alpha},\frac{t}{\epsilon^\beta}\right)}dB(t). \end{aligned} \tag{2.26}$$

Let $\mathbb{K}$ be a finite index set with $K = |\mathbb{K}|$. As a generalisation of the Helfrich elastic fluctuating membrane model of Section 2.2, we assume that the random field $H(x,t)$ can be written as $H(x,t) = h(x,\eta(t))$, where

$$h(x,\eta) = \sum_{k\in\mathbb{K}}\eta_k(t)e_k(x) = \langle\eta(t),e(x)\rangle,$$

where $e_k \in C^\infty(\mathbb{T}^d)$; these functions can be extended to $\mathbb{R}^d$ by periodicity. The stochastic process $\eta(t)$ is an $\mathbb{R}^K$-valued Ornstein-Uhlenbeck (OU) process given by

$$d\eta(t) = -\Gamma\eta(t)\,dt + \sqrt{2\Gamma\Pi}dW(t), \tag{2.27}$$

where $W(\cdot)$ is a standard $\mathbb{R}^K$-valued Brownian motion. The matrices Γ and Π are symmetric, positive definite. For simplicity, we shall assume that Γ and Π commute. This is to ensure that the OU process relaxes to equilibrium distribution defined by $\mathcal{N}(0,\Pi)$. This theory could be similarly developed without this assumption, but care must be taken to identify the correct invariant distribution. While this assumption is strong, it is sufficient for the models considered in this paper, and can be relaxed relatively easily.

Substituting this definition of $H(x,t)$ into (2.23), the evolution of the system can be described by the joint process $(X^\epsilon(t),\eta^\epsilon(t))$ is the solution of the following Itô SDE:

$$\begin{aligned} dX^\epsilon(t) &= \frac{1}{\epsilon^\alpha}F\left(\frac{X^\epsilon(t)}{\epsilon^\alpha},\eta^\epsilon(t)\right)dt + \sqrt{2\Sigma\left(\frac{X^\epsilon(t)}{\epsilon^\alpha},\eta^\epsilon(t)\right)}dB(t), \\ d\eta^\epsilon(t) &= -\frac{1}{\epsilon^\beta}\Gamma\eta^\epsilon(t)dt + \sqrt{\frac{2\Gamma\Pi}{\epsilon^\beta}}dW(t) \end{aligned} \tag{2.28}$$

where $F : \mathbb{T}^d \times \mathbb{R}^K \to \mathbb{R}^d$ is given by

$$F(x,\eta) := \frac{1}{\sqrt{|g|\,(x,\eta)}}\nabla\cdot\left(\sqrt{|g|}g^{-1}\right)(x,\eta), \tag{2.29}$$

$\Sigma : \mathbb{T}^d \times \mathbb{R}^K \to R^{2\times 2}_{\text{sym}}$ is

$$\Sigma(x,\eta) := g^{-1}(x,\eta), \tag{2.30}$$

and $g(x,\eta) := I + \nabla h(x,\eta) \otimes \nabla h(x,\eta)$. Since Γ and Π commute, it is straightforward to check that the OU process $\eta^\epsilon(t)$ is ergodic, with unique invariant measure given by

$$\mu_\eta = \mathcal{N}(0,\Pi), \tag{2.31}$$

with measure

$$\rho_\eta(\eta) \propto \exp\left(-\frac{\eta \cdot \Pi^{-1}\eta}{2}\right).$$

For simplicity we will assume that $\eta(0)$ is distributed according to ρ_η, so that the random field is started in stationarity.

Equivalently, one may consider the backward Kolmogorov equation corresponding to (2.28) for the evolution of an observable $u^\epsilon(x,\eta,t)$:

$$\begin{aligned} \frac{\partial u^\epsilon(x,\eta,t)}{\partial t} &= \mathcal{L}^\epsilon u^\epsilon(x,\eta,t), &&\text{for } (x,\eta,t) \in \mathbb{R}^d \times \mathbb{R}^K \times (0,T), \\ u^\epsilon(x,\eta,0) &= u(x,\eta), &&\text{for } (x,\eta) \in \mathbb{R}^d \times \mathbb{R}^K. \end{aligned} \tag{2.32}$$

The infinitesimal generator $\mathcal{L}^\epsilon$ can be written as

$$\mathcal{L}^\epsilon f(x,\eta) = \mathcal{L}_1^\epsilon f(x,\eta) + \mathcal{L}_2^\epsilon f(x,\eta).$$

The operator

$$\mathcal{L}_1^\epsilon f(x) := \frac{1}{\sqrt{|g|\,(x/\epsilon^\alpha,\eta)}} \nabla_x \cdot \left(\sqrt{|g|\,(x/\epsilon^\alpha,\eta)}\, g^{-1}(x/\epsilon^\alpha,\eta)\nabla_x f(x)\right),$$

encodes the effect of the rapid spatial fluctuations, while

$$\mathcal{L}_2^\epsilon f(\eta) := \frac{1}{\epsilon^\beta}\left(-\Gamma\eta \cdot \nabla_\eta + \Gamma\Pi : \nabla_\eta\nabla_\eta f\right),$$

describes the rapid temporal fluctuations.

The following proposition establishes the well-posedness of equation (2.28) for the joint process $(X^\epsilon(t), \eta^\epsilon(t))$.

Proposition 2.3.1. *Let X_0 and η_0 be random variables, independent of $B(\cdot)$ and $W(\cdot)$ such that $\mathbb{E}\,[X_0]^2 < \infty$ and $\mathbb{E}\,[\eta]^2 < \infty$. Then the system of SDEs (2.28) has a unique strong solution $(X^\epsilon(t), \eta^\epsilon(t))$ satisfying $X(0) = X_0$ and $\eta(0) = \eta_0$. Moreover, the solution $(X^\epsilon(t), \eta^\epsilon(t)) \in C([0,T];\mathbb{R}^d \times \mathbb{R}^K)$ is a Markov diffusion process.*

Proof. Without loss of generality we consider only $\epsilon = 1$. We apply the results of Sections 3.2 - 3.4 of [Friedman, 2006]. First we note that the drift and diffusion terms of (2.28) are smooth in X^ϵ and η, and so are locally Lipschitz. What remains to be verified is that the drift and diffusion coefficients have linear growth. The diffusion terms are clearly bounded since $\left|g^{-1}(x,\eta)\right|_2 \leq 1$, where $|\cdot|_2$ denotes the induced matrix 2-norm. We now consider the drift term $F(x,\eta)$ given by (2.29). Expanding, we have that

$$|F(x,\eta)| \leq \left|\frac{g^{-1}(x,\eta)\nabla_x\left(\sqrt{|g|\,(x,\eta)}\right)}{\sqrt{|g|\,(x,\eta)}}\right| + \left|\nabla_x \cdot g^{-1}(x,\eta)\right|, \tag{2.33}$$

where for $v \in \mathbb{R}^d$, $|v|$ denotes the Euclidean norm in $\mathbb{R}^d$. The first term satisfies

$$\begin{aligned}\left|\frac{g^{-1}(x,\eta)\nabla_x\left(\sqrt{|g|\,(x,\eta)}\right)}{\sqrt{|g|\,(x,\eta)}}\right| &\leq \left|\nabla_x\left(\sqrt{|g|\,(x,\eta)}\right)\right| \\ &\leq \frac{\left|\left(\nabla_x\nabla_x h\right)\nabla_x h\right|}{\sqrt{|g|\,(x,\eta)}} \\ &\leq \left|\nabla_x\nabla_x h\right|_2 \\ &\leq CK\,|\eta|,\end{aligned} \tag{2.34}$$

for some positive constant C, where we used (2.2) in the third inequality, and the fact that $h(x,\eta) = \eta\cdot e(x)$, the components $e(x)$ and all their derivatives are bounded to obtain the final inequality. Similarly,

$$\nabla_x \cdot g^{-1}(x,\eta) = -2\left(g^{-1}(x,\eta) : \nabla_x\nabla_x h(x,\eta)\right)\left(g^{-1}(x,\eta)\nabla_x h(x,\eta)\right).$$

Noting that

$$g^{-1}(x,\eta)\nabla_x h(x,\eta) = \frac{1}{|g|\,(x,\eta)}\nabla_x h(x,\eta),$$

it follows that

$$\left|\nabla_x \cdot g^{-1}(x,\eta)\right| \leq 2\left|g^{-1}\right|_F \left|\nabla_x\nabla_x h(x,\eta)\right|_F \leq C'K\,|\eta|, \tag{2.35}$$

for some constant $C' > 0$, where $|\cdot|_F$ denotes the Frobenius matrix norm. Together, (2.34) and (2.35) along with the linearity of $\Gamma\eta$ show that

$$\left|\begin{pmatrix} F(x,\eta) \\ -\Gamma\eta \end{pmatrix}\right| \leq C''K\,|\eta|,$$

for some constant C'' independent of x and η. The conclusion of the lemma now follows by Theorems 2.2, 3.6 and 4.2 of [Friedman, 2006]. $\square$

As before, we wish to study the behaviour of $X^\epsilon(t)$ and it's corresponding backward Kolmogorov equation as $\epsilon \to 0$. Since the parameters α and β quantify the relative speed of the spatial and temporal fluctuations, respectively, we expect that the limiting behaviour will vary for different values of α and β. In Chapter 5 we will study the scaling limits of (2.28) for the following four scaling regimes.

Case I: $\alpha = 1$ **and** $\beta = -\infty$
In this regime the temporal fluctuations occur on a timescale slower than the characteristic timescale of the diffusion process, so far the model can be considered to be a diffusion on a stationary realisation of the surface. This situation has been studied in the case of diffusion on a quenched Helfrich elastic membrane in [Naji and Brown, 2007]. This regime is studied as the problem of diffusion on a static surface with rapid spatial oscillations.

Case II: $\alpha = 0$ **and** $\beta = 1$
The microscopic fluctuations are contributed entirely by the temporal fluctuations. The motivating example in this regime is that of diffusion on an Helfrich elastic membrane; this problem has been studied in great detail (see [Naji and Brown, 2007; Gustafsson and Halle, 1997; Reister and Seifert, 2007]).

Case III: $\alpha = 1$ **and** $\beta = 1$
In this regime we consider lateral diffusion on surfaces possessing both rapid spatial and temporal fluctuations, with the spatial and temporal fluctuations occurring at comparable scales. While this regime has not been studied before, it naturally extends the work covered in [Halle and Gustafsson, 1997; Naji and Brown, 2007; Reister and Seifert, 2007] and helps provide a complete picture. This regime is interesting due to the fact that the limiting diffusion coefficient is in some sense intermediate between that of Case I and Case II.

Case IV: $\alpha = 1$ **and** $\beta = 2$
In this regime we consider surfaces with both rapid spatial and temporal fluctuations but the temporal fluctuations occur at a faster scale compared to the spatial fluctuations. As in Case III, this regime has not been considered previously, but is studied to provide a complete picture of the possible limiting behaviour.

As we will show in Chapter 5, the limiting behaviour of the process will exhibit different behaviour, in particular the effective diffusion will be distinct in each case. For example, if we consider the effective diffusion coefficient for the Helfrich elastic fluctuating model, then in the Case II regime, we can show that the (non-dimensional) effective diffusion will converge to $\frac{1}{2}$ as the ultraviolet cutoff c converges to infinity. On the other hand, in the Case III regime, the effective diffusion coefficient will converge to 0.

The problem in the Case I regime reduces to the problem of lateral diffusion on a static, periodic surface with rapid undulations. Thus we can apply the results of Chapter 3 to identify the limiting equations, and study the properties of the effective diffusion. In this case, due to the lack of temporal fluctuations, the effective diffusion coefficient will be dependent on the particular realisation of the random field

$H(x,t)$. As was considered in previous work ([Naji and Brown, 2007] and [Reister and Seifert, 2007]), studying the properties of the effective diffusion coefficient averaged over the surface realisations is more illuminating.

In Case II, the limiting behaviour is determined by the properties of the stationary distribution of the Ornstein Uhlenbeck process $\eta(t)$ and deriving the effective diffusion process can be viewed as an averaging problem (see [Pavliotis and Stuart, 2008]).

In the regimes covered by Case III and Case IV we must consider the interactions between the temporal and spatial fluctuations. In Case III, the spatial fluctuations homogenise the diffusion process “faster" than the temporal fluctuations, and the result is that the effective diffusion coefficient will merely be the effective diffusion coefficient from Case I averaged over the invariant measure of the OU process $\eta(t)$. SDEs in this scaling limit were considered in [Garnier, 1997].

Deriving a scaling limit in the Case IV regime proves more complicated, due to the lack of an explicit invariant measure for the “fast process". Once the geometric ergodicity of the fast process with respect to a unique invariant measure is established, the approach will be similar to the classical probabilistic homogenisation arguments of [Bensoussan et al., 1978]. Although a limiting equation is established, the lack of an explicit invariant measure, makes it hard to establish bounds on the effective diffusion coefficient.

We have not yet addressed the question of the limiting behaviour of $X^{\epsilon}(t)$ for other values of α and β besides those considered in Cases I - IV. The answer to this question is problem dependent (i.e. dependent on the particular choice of $e(x)$, and the OU process drift and diffusion coefficients, Γ and Π). However, in Chapter 6, for the particular case of diffusion on a two-dimensional thermally fluctuating Helfrich surface we will show that the limits corresponding to Case I to Case IV are exhaustive, in this sense that these are the only distinguished limits that can arise from this system.

Chapter 3

DIFFUSION ON A STATIC SURFACE WITH PERIODIC FLUCTUATIONS

In this chapter we study the macroscopic behaviour of particles diffusing laterally on a static surface possessing rapid, periodic fluctuations as described in Section 2.3.1. While this time-independent model is of limited use as a phenomenological model, it benefits from being amenable to classical periodic homogenisation methods and will serve as the basis for the more elaborate models considered in subsequent chapters. The problem of diffusion on static, periodically fluctuating surfaces has been previously studied in the literature with varying degrees of rigour. A review of existing work on this problem is given in Section 3.1. While the homogenised equations and the variational bounds for the effective diffusion coefficient have been previously stated (in particular [Halle and Gustafsson, 1997]), to our knowledge these results have never been analysed rigorously as we do here.

We recapitulate the details of Section 2.3.1. Consider the projected trajectory $X^\epsilon(t)$ of a particle diffusing laterally on the surface with Monge gauge

$$h^\epsilon(x) = \epsilon h\left(\frac{x}{\epsilon}\right), \tag{3.1}$$

where $\epsilon > 0$ is a small scale parameter and h is a sufficiently smooth real-valued function on $\mathbb{R}^d$ which is periodic with periodic derivatives with period 1 in all directions. The particle's position satisfies the following Itô SDE

$$dX^\epsilon(t) = \frac{1}{\epsilon}F(X(t)/\epsilon)\,dt + \sqrt{2\Sigma(X(t)/\epsilon)}\,dB(t), \tag{S1}$$

where

$$F(x) = \frac{1}{\sqrt{|g|\,(x)}}\nabla\cdot\left(\sqrt{|g|\,(x)}g^{-1}(x)\right),$$

and

$$\Sigma(x) = g^{-1}(x),$$

where $g(x)$ is the metric tensor of the surface with Monge gauge $h(x)$, i.e.

$$g(x) = I + \nabla h(x) \otimes \nabla h(x).$$

The backward Kolmogorov equation corresponding to (S1), for an observable

$$u^\epsilon(x,t) = \mathbb{E}\left[u(X^\epsilon(t) \,|\, X^\epsilon(0) = x\right],$$

where $u \in C_b(\mathbb{R}^d)$ is given by:

$$\begin{aligned} \frac{\partial u^\epsilon(x,t)}{\partial t} &= \mathcal{L}^\epsilon u^\epsilon(x,t), \qquad (x,t) \in \mathbb{R}^d \times (0,T], \\ u^\epsilon(x,t) &= u(x), \qquad\qquad (x,t) \in \mathbb{R}^d \times \{0\}. \end{aligned} \tag{P1}$$

where

$$\mathcal{L}^\epsilon f(x) = \frac{1}{\sqrt{|g|\,(x/\epsilon)}} \nabla_x \cdot \left(\sqrt{|g|\,(x/\epsilon)} g^{-1}(x/\epsilon) \nabla_x f(x) \right). \tag{3.2}$$

Our objective is to study the effective behaviour of $X^\epsilon(t)$ and $u^\epsilon(x,t)$ as $\epsilon \to 0$. We will show that as $\epsilon \to 0$, the $\mathbb{R}^d$-valued process $X^\epsilon(t)$ will converge weakly to a Brownian motion on $\mathbb{R}^d$ with constant diffusion coefficient D which depends on the surface map $h(x)$. Equivalently, we show that u^ϵ converges pointwise to the solution u^0 of the PDE:

$$\begin{aligned} \frac{\partial u^0(x,t)}{\partial t} &= D : \nabla_x \nabla_x u^0(x,t), \qquad (x,t) \in \mathbb{R}^d \times (0,T], \\ u^0(x,t) &= v(x), \qquad\qquad (x,t) \in \mathbb{R}^d \times \{0\}. \end{aligned} \tag{3.3}$$

Since (S1) (respectively (P1)) is a SDE (resp. PDE) with periodic coefficients, the problem is amenable to classical periodic homogenisation methods, such as those of [Bensoussan et al., 1978; Jikov et al., 1994; Pavliotis and Stuart, 2008]. In Section 3.2 we state the homogenisation result for this model. The result will be justified formally by using perturbation expansions of the PDE in(P1). One can then invoke standard results to obtain a rigorous proof of convergence for both (S1) and (P1).

For $d = 1$, D depends only on the excess surface area Z (i.e. the average ratio of the surface area of the graph of h to the area of the base). Indeed, we will show that $D = \frac{1}{Z^2}$. For $d \geq 2$, in general, the expression for D depends of a *corrector*, the solution of an auxiliary *cell* problem, which has no closed-form solution in general. Without making further assumptions on the surface we can obtain at best upper and lower bounds in terms of the function h which are derived in Section 3.3. In the special case where $d = 2$ and D is isotropic, however, it is possible to obtain a closed form expression for the effective diffusion coefficient, and in Section 3.4, making use of a duality transformation argument we are able to show that D is equal to $\frac{1}{Z}\mathbf{I}$. The expression $D_{as} = \frac{1}{Z}\mathbf{I}$, which measures the ratio of the projected surface area to the curvilinear surface area of the curved surface with respect to the plane, is known in the literature as the *area scaling approximation*, [Halle and Gustafsson, 1997; Naji and Brown, 2007; King, 2004; Gov, 2006].

The rest of the chapter is organised as follows. In Section 3.5 we identify a natural symmetry condition for the surface map h which is sufficient to ensure that D is isotropic, and thus that the area scaling approximation holds. In Section 3.6 we describe a finite element scheme for numerically approximating D, and in Section 3.7 we use this scheme to perform numerical experiments illustrating the theory of this chapter. Finally, in Section 3.8 we use the results obtained in this chapter to study the effective behaviour of (2.28) in the Case I regime. Although the effective diffusion coefficient will depend on the particular realisation of the surface, we are still able to obtain useful information regarding the average effective diffusion coefficient (i.e. the effective diffusion coefficient averaged over all realisations). We then focus on the specific case of diffusion on a quenched Helfrich elastic surface and study how the limiting behaviour is affected by the model parameters, both analytically and with numerical experiments.

3.1 PREVIOUS WORK

Lateral diffusion on quasi-planar periodic surfaces have been previously studied in the literature, mainly in the context of biological interfaces. The first such work we are aware of is [Aizenbud and Gershon, 1985] where the authors consider the problem of diffusion on a curve possessing rapid periodic fluctuations, with the objective of explaining the slowing down of diffusion of succiny-concanavalin A receptors on the surfaces of adherent mouse fibroblast. The authors derive the effective diffusion coefficient, in this case, given by $D = \frac{1}{Z^2}$. In [Halle and Gustafsson, 1997], the authors study the same problem in two dimensions. By recognizing the problem as diffusion in a potential they use standard results from diffusions in periodic potentials to obtain the homogenised diffusion coefficient D in terms of a corrector. Under some implicit symmetry assumptions on the surface, they then derive variational bounds for D. The authors discuss various non-variational bounds for the D, and propose two estimates: the effective medium approximation, given by

$$D_{ema} = \frac{Z}{\int_{\mathbb{T}^d} |g|\,(y)\,dy}\mathbf{I},$$

and the area scaling approximation, given by

$$D_{as} = \frac{1}{Z}\mathbf{I}.$$

The authors claim that D_{ema} is the better approximation (which is at odds with our conclusion), and provide a very heuristic justification for its validity, refining a previous result of [Sokolov, 1987]. In [King, 2004], the author studies the problem of diffusion on a quasi-planar surface defined by a periodically repeated crested cycloid. The effective diffusion coefficient was computed numerically by simulating random walks on the surface and estimating the diffusion coefficient using linear regression. The conclusion was that $D \propto Z^{-1.42}$, which is at odds with the results presented in here and in [Halle and Gustafsson, 1997].

In [Naji and Brown, 2007], the authors also study the problem of diffusion on a surface given by a realisation of a quenched elastic membrane with periodic fluctuations. By performing direct simulation of the Brownian particles over realisations of the surface, the authors estimate the effective diffusion coefficient, averaged over the surface realisations. Based on these numerical estimates, the authors conclude that for static surfaces the area scaling estimate is the most accurate, but do not offer a proof.

More recently, in [Khrabustovskyi, 2009], the author studies the asymptotic behaviour of the spectra and eigenfunctions of the Laplace-Beltrami operator of a (non-graph) manifold consisting of a bounded two-dimensional domain where small disjoint holes of size $O(\epsilon)$, placed periodically with period $O(\epsilon)$, are removed and replaced with "bubbles" - n-dimensional spheres, truncated in a region about the poles. The author then applies the results derived for the spectra to study the asymptotic behaviour of the heat equation on such a surface, and proves convergence of the solution to a diffusion on the plane with effective diffusion coefficient depending on the (possibly slowly varying) radii of the holes.

3.2 HOMOGENIZATION RESULT

For convenience, we introduce the fast process $Y(t) = \frac{X(t)}{\epsilon}\ \mathbf{mod}\ \mathbb{T}^d$. We can then express (S1) as the following fast-slow system

$$\begin{aligned} dX^\epsilon(t) &= \frac{1}{\epsilon}F(Y^\epsilon(t))\,dt + \sqrt{2\Sigma(Y^\epsilon(t)}\,dB(t), \\ dY^\epsilon(t) &= \frac{1}{\epsilon^2}F(Y^\epsilon(t))\,dt + \sqrt{\frac{2}{\epsilon^2}\Sigma(Y^\epsilon(t))}\,dB(t), \end{aligned} \tag{3.4}$$

where $X^\epsilon(t) \in \mathbb{R}^d$, $Y^\epsilon(t) \in \mathbb{T}^d$ and $B(t)$ is a standard Brownian motion on $\mathbb{R}^d$. The infinitesimal generator of the fast process is the $L^2(\mathbb{T}^d)$ closure of

$$\mathcal{L}_0 f(y) = \frac{1}{\sqrt{|g|\,(y)}}\nabla_y \cdot \left(\sqrt{|g|\,(y)}g^{-1}(y)\nabla_y f(y)\right), \quad f \in C^2(\mathbb{T}^d).$$

It is straightforward to see that $\mathcal{L}_0$ is a uniformly elliptic operator with nullspace containing only constants, that is

$$\mathcal{N}[\mathcal{L}_0] = \{\mathbf{1}\},$$

and

$$\mathcal{N}[\mathcal{L}_0^*] = \{\rho(y)\},$$

where $\rho(y) = \frac{\sqrt{|g|(y)}}{Z}$, and where Z is the normalisation constant given by $Z = \int_{\mathbb{T}^d} \sqrt{|g|\,(y)}\, dy$.

We expect to be able to compute the homogenising effect of the fast process Y^ϵ on the slow process X^ϵ and thereby compute an effective equation which accounts for, but removes explicit reference to, the small scale. Given $v \in C_b^2(\mathbb{R}^2 \times \mathbb{T}^d)$, the observable

$$v^\epsilon(x, y) := \mathbb{E}\left[v\left(X^\epsilon(t), Y^\epsilon(t)\right) \mid X^\epsilon(0) = x, Y^\epsilon(0) = \frac{x}{\epsilon}\right]$$

satisfies the backward Kolmogorov equation given by:

$$\frac{\partial v^\epsilon(x, y, t)}{\partial t} = \mathcal{L}^\epsilon v^\epsilon(x, y, t), \quad (x, y, t) \in \mathbb{R}^d \times \mathbb{T}^d \times (0, T], \tag{3.5}$$

where

$$\mathcal{L}^\epsilon = \mathcal{L}_2 + \frac{1}{\epsilon}\mathcal{L}_1 + \frac{1}{\epsilon^2}\mathcal{L}_0 \tag{3.6}$$

for

$$\begin{aligned} \mathcal{L}_1 v(x, y) :=& F(y) \cdot \nabla_x v(x, y) \\ &+ 2\Sigma(y) : \nabla_x \nabla_y v(x, y), \end{aligned} \tag{3.7}$$

and

$$\mathcal{L}_2 v(x, y) := \Sigma(y) : \nabla_x \nabla_x v(x, y). \tag{3.8}$$

Note that the last term in (3.7) reflects the correlation of the noise between the fast and slow processes.

We wish to study the behaviour of X^ϵ and v^ϵ in the limit as $\epsilon \to 0$, homogenising over the fast variable Y^ϵ to identify a constant coefficient diffusion equation which approximates the slow process in some sense. As the corresponding SDE and PDE have periodic coefficients, we can apply results from classical homogenisation theory such as [Bensoussan et al., 1978; Jikov et al., 1994] to prove convergence of X^ϵ and v^ϵ to solutions of limiting equations. We refer to [Pavliotis and Stuart, 2008] for a modern pedagogical treatment of this theory. In this Section we will state the homogenisation result and provide a formal derivation of the limiting equations based on perturbation expansions.

The macroscopic effect of the fast-scale fluctuations is characterised by a *corrector* $\chi : \mathbb{T}^d \to \mathbb{R}^d$ which is the solution of the following *cell equation*

$$\mathcal{L}_0 \chi(y) = -F(y), \qquad y \in \mathbb{T}^d. \tag{3.9}$$

Lemma 3.2.1. *There exists a unique solution $\chi \in C^2(\mathbb{T}^d; \mathbb{R}^d)$ such that*

$$\int_{\mathbb{T}^d} \chi(y)\rho(y)\, dy = 0, \tag{3.10}$$

and which solves (3.9).

Proof. Since $\mathcal{L}_0$ has a compact resolvent, we can apply the Fredholm alternative [Gilbarg and Trudinger, 2001]. This states that the cell equation will have a solution provided the RHS of (3.9) is orthogonal in $L^2(\mathbb{T}^d)$ to the nullspace of $\mathcal{L}_0^*$, that is, the *centering condition* holds

$$\int_{\mathbb{T}^d} F(y)\,\rho(y)dy = \mathbf{0},$$

but is trivially true from the definition of $F(y)$ and $\rho(y)$. The solution χ is unique provided we impose condition (3.10). Finally, $\chi \in C^2(\mathbb{T}^d;\mathbb{R}^d)$ follows from standard elliptic regularity theory. □

The following theorem states the homogenisation result for this scaling regime. The proof given is based on formal perturbation expansions which can be used as the basis for a rigorous proof. However, a probabilistic approach based on Theorem 3.1, [Pardoux, 1999] or [Bensoussan et al., 1978] are more succint. In what follows we will adopt the convention that

$$\left(\nabla_y \chi(y)\right)_{ij} = \frac{\partial \chi_i}{\partial y_j}(y), \quad \text{for } i,j \in 1,\dots,d$$

see Chapter 2 of [Gonzalez and Stuart, 2008].

Theorem 3.2.2. *Let $T > 0$, then the process X^ϵ converges weakly in $C([0,T];\mathbb{R}^d)$ to a Wiener process $X^0(t)$ which solves*

$$dX^0(t) = \sqrt{2D}dB(t), \tag{3.11}$$

where D is the (constant) effective diffusion coefficient given by

$$D = \frac{1}{Z}\int_{\mathbb{T}^d} \left(I + \nabla_y\chi(y)\right) g^{-1}(y) \left(I + \nabla_y\chi(y)\right)^\top \sqrt{|g|\,(y)}\,dy, \tag{3.12}$$

where Z is the surface area of a single cell of the surface given by

$$Z = \int_{\mathbb{T}^d} \sqrt{|g|\,(y)}\,dy. \tag{3.13}$$

Moreover, if equation (P1) has initial data u (independent of ϵ) such that $u \in C_b^2(\mathbb{R}^d)$, then the solution u^ϵ of (P1) converges pointwise to the solution u_0 of (3.3) uniformly with respect to t over $[0,T]$.

Formal justification of Theorem 3.2.2. To derive the homogenised equation in this regime we make the ansatz that the solution v^ϵ of the backward Kolmogorov equation (3.5) is of the form

$$v^\epsilon(x,y,t) = v_0(x,y,t) + \epsilon v_1(x,y,t) + \epsilon^2 v_2(x,y,t) + \dots, \tag{3.14}$$

for smooth $v_i : \mathbb{R}^d \times \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$. Substituting (3.14) in (3.5) and identifying equal powers of ϵ, we obtain the following equations

$$\mathcal{L}_0 v_0(x,y,t) = 0 \quad (x,y,t) \in \mathbb{R}^d \times \mathbb{T}^d \times (0,T] \tag{3.15}$$

$$\mathcal{L}_0 v_1(x,y,t) = -\mathcal{L}_1 v_0 \quad (x,y,t) \in \mathbb{R}^d \times \mathbb{T}^d \times (0,T] \tag{3.16}$$

$$\mathcal{L}_0 v_2(x,y,t) = \frac{\partial v_0}{\partial t} - \mathcal{L}_1 v_1 - \mathcal{L}_2 v_0 \quad (x,y,t) \in \mathbb{R}^d \times \mathbb{T}^d \times (0,T]. \tag{3.17}$$

Since the nullspace of $\mathcal{L}_0$ contains only constants in y, equation (3.15) implies that v_0 is a function of x and t only. Equation (3.16) becomes

$$\mathcal{L}_0 v_1(x,y,t) = -F(y) \cdot \nabla_x v_0(x,t). \tag{3.18}$$

Let $\chi \in C^2(\mathbb{T}^d; \mathbb{R}^d)$ be the solution of the cell equation (3.9). If we choose $v_1 = \chi \cdot \nabla_x v_0(x,t)$, then it is clear that v_1 solves (3.16).

Finally, by the Fredholm alternative on $\mathcal{L}_0$, a necessary condition for equation (3.17) to have a solution is that the RHS of (3.17) has mean zero with respect to the measure ρ, that is,

$$\frac{\partial v_0}{\partial t} \int_{\mathbb{T}^d} \rho(y)\, dy = \int_{\mathbb{T}^d} \mathcal{L}_1 v_1 \rho(y)\, dy + \int_{\mathbb{T}^d} \mathcal{L}_2 v_0 \rho(y)\, dy.$$

Substituting v_0 and v_1, we obtain

$$\begin{aligned}\frac{\partial v_0}{\partial t} &= \frac{1}{Z} \int_{\mathbb{T}^d} \nabla_y \cdot \left(\sqrt{|g|\,(y)} g^{-1}(y) \right) \cdot \nabla_x \left(\chi \cdot \nabla_x v_0 \right)\, dy \\ &+ \frac{2}{Z} \int_{\mathbb{T}^d} \sqrt{|g|\,(y)} g^{-1}(y) : \nabla_x \nabla_y \left(\chi \cdot \nabla_x v_0 \right)\, dy \\ &+ \frac{1}{Z} \int_{\mathbb{T}^d} \sqrt{|g|\,(y)} g^{-1}(y) : \nabla_x \nabla_x v_0(x)\, dy.\end{aligned}$$

Integrating the first term by parts with respect to y and simplifying, we obtain:

$$\frac{\partial v_0}{\partial t} = \left(\frac{1}{Z} \int_{\mathbb{T}^d} g^{-1}(y) \big(I + \nabla_y \chi(y) \big)^\top \sqrt{|g|\,(y)}\, dy \right) : \nabla_x \nabla_x v_0.$$

Thus, the homogenised diffusion equation for v_0 is

$$\frac{\partial v_0}{\partial t} = D : \nabla_x \nabla_x v_0, \tag{3.19}$$

where

$$D = \frac{1}{Z} \int_{\mathbb{T}^d} g^{-1}(y) \big(I + \nabla_y \chi(y) \big)^\top \sqrt{|g|\,(y)}\, dy.$$

Multiplying (3.9) by $\chi(y)\rho(y)$ and integrating by parts gives:

$$\int_{\mathbb{T}^d} \left(I + \nabla_y \chi(y)\right) g^{-1}(y) \nabla_y \chi(y)^\top \sqrt{|g|\,(y)}\, dy = 0,$$

so that the effective diffusion coefficient can be written in the following symmetric form

$$D = \frac{1}{Z} \int_{\mathbb{T}^d} \big(I + \nabla_y \chi(y)\big) g^{-1}(y) \big(I + \nabla_y \chi(y)\big)^\top \sqrt{|g|\,(y)}\, dy.$$

Noting that (3.19) is the backward Kolmogorov equation for the solution of (3.11), this therefore provides a formal justification that the process X^ϵ converges in law to X^0. □

An equation equivalent to (3.12) for D had already been derived in [Gustafsson and Halle, 1997] under symmetry assumptions on the spatial fluctuations. They obtained this expression by translating results in [Jackson and Coriell, 1963] and [Festa and d'Agliano, 1978], which consider the homogenisation of diffusion in a periodic potential, to the problem of diffusion on a static membrane with periodic fluctuations.

In the one-dimensional case, by integrating (3.9) directly, we see that the corrector χ satisfies

$$\frac{d\chi}{dy}(y) + 1 = \frac{\sqrt{|g|\,(y)}}{Z}.$$

Substituting this into (3.12), we obtain

$$D = \frac{1}{Z^2},$$

so that the homogenised equation (3.3) becomes

$$\frac{\partial u_0(x,t)}{\partial t} = \frac{1}{Z^2} \Delta u_0(x,t), \quad (x,t) \in \mathbb{R}^1 \times (0,T]. \tag{3.20}$$

The value of D agrees with the value derived heuristically in the discussion following equation (2.17) in Chapter 2.

3.3 PROPERTIES OF THE EFFECTIVE DIFFUSION COEFFICIENT

In two-dimensions or higher it is not, in general, possible to solve for the corrector χ explicitly and thus D has no closed form. However, we can identify certain properties of the effective diffusion coefficient. Let

$$H^1_{per}(\mathbb{T}^d) := \left\{ v \in H^1(\mathbb{T}^d) \,\middle|\, \int_{\mathbb{T}^d} v(y)\, dy = 0 \right\},$$

and $S^d = \{e \in \mathbb{R}^{d+1} \,|\, |e| = 1\}$. The following proposition illustrates the basic properties of D, valid in all dimensions.

Proposition 3.3.1. *Let $e \in S^{d-1}$, then*

1. *D is a symmetric, positive definite matrix.*
2. *D can be characterised via the expression*

$$e \cdot De = \inf_{v \in H^1_{per}(\mathbb{T}^d)} L[v, e], \tag{3.21}$$

where

$$L[v, e] := \frac{1}{Z} \int_{\mathbb{T}^d} (e + \nabla v(y)) \cdot g^{-1}(y)\, (e + \nabla v(y))\, \sqrt{|g|\,(y)}\, dy$$

and $\chi^e = \sum_{i=1}^d \chi_i e_i$ is the unique minimiser of (3.21).

3. *The following Voigt-Reuss bounds [Jikov et al., 1994, Section 1.6] hold,*

$$e \cdot D_* e \le e \cdot De \le e \cdot D^* e$$

where

$$D_* = \frac{1}{Z} \left(\int_{\mathbb{T}^d} \frac{g(y)}{\sqrt{|g|\,(y)}}\, dy \right)^{-1}, \tag{3.22}$$

and

$$D^* = \frac{1}{Z} \left(\int_{\mathbb{T}^d} g^{-1}(y) \sqrt{|g|\,(y)}\, dy \right). \tag{3.23}$$

4. *In particular, the homogenised diffusion coefficient D satisfies the following inequality,*

$$e \cdot De \le 1.$$

5. *Finally, the effective diffusion coefficient can be expressed as follows*

$$\begin{aligned} e \cdot De = &\frac{1}{Z} \int_{\mathbb{T}^d} e \cdot g^{-1}(y) e \sqrt{|g|\,(y)}\, dy \\ &- \frac{1}{Z} \int_{\mathbb{T}^d} \nabla_y \chi^e(y) \cdot g^{-1}(y) \nabla_y \chi^e(y)\, \sqrt{|g|\,(y)}\, dy. \end{aligned} \tag{3.24}$$

Remark Since the microscopic diffusion coefficient in the nondimensionalized equation is I, Property (4) implies that the microscopic spatial fluctuations deplete the diffusion at the macroscopic level, so that effective diffusion is depleted with respect to the molecular diffusion coefficient.

Proof. Property (2) follows by noting that the Euler-Lagrange equation for the minimiser of (3.21) is given by (3.9) which has a unique solution $\chi \cdot e$ in $H^1_{per}(\mathbb{T}^d)$. Moreover, $e \cdot De = L[\chi^e, e]$. It follows that for each unit vector $e \in \mathbb{R}^d$,

$$e \cdot D\, e \leq L[0,e] = \frac{1}{Z} e \cdot \left(\int_{\mathbb{T}^2} g^{-1}(y) \sqrt{|g|\,(y)}\, dy \right) e =: e \cdot D^* e,$$

proving the second inequality of (3). To derive the Voigt-Reuss type lower bound [Jikov et al., 1994] in (3) we note that for fixed $e \in S^{d-1}$,

$$e \cdot D_* e := \inf_{\substack{\Phi \in L^2(\mathbb{T}^d)^d \\ \int_{\mathbb{T}^d} \Phi(y)\, dy = 0}} \frac{1}{Z} \left(e + \Phi(y)\right) \cdot g^{-1}(y) \left(e + \Phi(y)\right) \sqrt{|g|\,(y)}\, dy \leq e \cdot D\, e.$$

The corresponding Euler-Lagrange equation is given by

$$g^{-1}(y) \left(e + \Phi(y)\right) \sqrt{|g|\,(y)} = C,$$

where C is a Lagrange multiplier for the constraint $\int_{\mathbb{T}^d} \Phi(y)\, dy = 0$, this can be solved explicitly to show that

$$\Phi(y) + e = \frac{g(y)}{\sqrt{|g|\,(y)}} \left(\int_{\mathbb{T}^d} \frac{g(y)}{\sqrt{|g|\,(y)}} \right)^{-1} e,$$

so that

$$e \cdot D_* e = \frac{1}{Z} e \cdot \left(\int_{\mathbb{T}^2} \frac{g(y)}{\sqrt{|g|\,(y)}}\, dy \right)^{-1} e,$$

thus proving (3). Moreover, the positive-definiteness of D follows immediately from that of D_*. Using the fact that $\left|g^{-1}\right| \leq 1$, it follows that

$$e \cdot De \leq e \cdot D^* e \leq 1,$$

and thus proving (4). The symmetry of D follows from the symmetry of the inverse metric tensor, proving (1).

To prove (5), we expand expression (3.12) for $e \cdot De$ to get:

$$\begin{aligned} e \cdot De = &\frac{1}{Z} \int_{\mathbb{T}^d} e \cdot g^{-1}(y)\, e \sqrt{|g|\,(y)}\, dy \\ &+ \frac{2}{Z} \int_{\mathbb{T}^d} e \cdot g^{-1}(y) \nabla_y \chi^e(y) \sqrt{|g|\,(y)}\, dy \\ &+ \frac{1}{Z} \int_{\mathbb{T}^d} \nabla_y \chi^e(y) \cdot g^{-1}(y) \nabla_y \chi^e(y) \sqrt{|g|\,(y)}\, dy, \end{aligned} \tag{3.25}$$

where we use the symmetry of $g^{-1}(y)$ to obtain the second line. Now, multiplying the cell equation (3.41) by χ^e and integrating by parts over $\mathbb{T}^d$ we obtain that

$$\int_{\mathbb{T}^d} \nabla_y \chi^e \cdot g^{-1}(y) \nabla_y \chi^e \sqrt{|g|(y)}\, dy = -\int_{\mathbb{T}^d} e \cdot g^{-1}(y) \nabla_y \chi^e(y) \sqrt{|g|(y)}\, dy. \quad (3.26)$$

Substituting for the second line in (3.25) we obtain the desired expression. □

Using expression (3.21) for the macroscopic diffusion D it is possible to obtain sharper upper bounds than D^* by minimising over a proper closed subset of $H^1_{per}(\mathbb{T}^d)$. For example, minimising $L[v, e]$ over the subset which varies only along a single dimension, then it is possible to obtain the following upper bound for diffusions along the coordinate directions e_i, $i = 1, \dots d$,

$$e_i \cdot D e_i \leq e_i \cdot D^+ \cdot e_i \leq \frac{1}{Z} \int_{\mathbb{T}^d} e_i \cdot g^{-1}(y) e_i \sqrt{|g|(y)}\, dy,$$

for D^+_{ij} given by:

$$D^+_{ij} := \frac{1}{Z} \left(\int_0^1 \frac{1}{M_i \left[g^{ii}(y) \sqrt{|g|(y)} \right]}\, dy_1 \right)^{-1}, \qquad i = j,$$

$$D^+_{ij} := 0, \qquad i \neq j,$$

and where $M_i[f]$ is the marginal of f with respect to the variable y_i. Although generally better than D^*, the above upper bound will not be sharp for general periodic surfaces.

3.4 THE AREA SCALING APPROXIMATION

In this section, restricting our attention to the two dimensional case, we make use of a basic duality transformation result to derive tight bounds on the eigenvalues of the effective diffusion coefficient. Moreover, in the case where the effective diffusion is isotropic we exhibit an explicit expression for the effective diffusion coefficient, depending only on Z. In their simplest forms, duality transformations provide a means of relating the effective diffusion coefficient σ_* obtained through homogenising an elliptic PDE with a rapidly varying matrix σ to the effective diffusion coefficient σ'_* of a dual problem, obtained through a $90°$ rotation about the origin. The existence of such a duality depends strongly on the fact that in two dimensions the $90°$ rotation of a divergence-free field is curl-free and vice versa. Such transforms were used firstly in conductance problems by [Keller, 1963], and subsequently by [Matheron, 1967; Dykhne, 1971; Mendelson, 1975; Kohler and Papanicolaou, 1982] and several others. The form of the duality transformation we present here is equivalent to that of [Jikov et al., 1994, Section 1.5], although the proof is different.

Proposition 3.4.1. *For $d = 2$, D satisfies the following relationship*

$$\det(D) = \frac{1}{Z^2}. \tag{3.27}$$

In particular, if D is isotropic, then D can be written explicitly as

$$D = \frac{1}{Z}\mathbf{I}.$$

Proof. We follow the approach used in [Kohler and Papanicolaou, 1982]. Define the following two sets of admissible functions.

$$\mathcal{F} = \left\{ F \in \left(H^1_{per}(\mathbb{T}^2)\right)^2, \middle| \nabla \cdot F = 0, \int F = 0 \right\},$$

and

$$\mathcal{G} = \left\{ F \in \left(H^1_{per}(\mathbb{T}^2)\right)^2, \middle| \nabla \times G = 0, \int G = 0 \right\}.$$

By Thompson's duality principle, [Mei and Vernescu, 2010, Section 2.6.2], we have the following relation

$$e \cdot (ZD)^{-1} e = \inf_{F \in \mathcal{F}} \int_{\mathbb{T}^2} (F + e) \cdot \frac{g(y)}{\sqrt{|g|(y)}} (F + e)\, dy \tag{3.28}$$

Let $Q : \mathbb{R}^2 \to \mathbb{R}^2$ denote a $\frac{\pi}{2}$ rotation about the origin in $\mathbb{R}^2$. It is straightforward to see that in two dimensions, the map $\mathcal{Q} : \mathcal{G} \to \mathcal{F}$ defined by

$$\mathcal{Q}G(y) = QF(y),$$

is a bijection between the two sets. In particular, substituting $F = QG$ in (3.28), we obtain that

$$\begin{aligned} e \cdot (Z\,D)^{-1} e &= \inf_{G \in \mathcal{G}} \int_{\mathbb{T}^2} (QG + e) \cdot \frac{g(y)}{\sqrt{|g|(y)}} (QG + e)\, dy \\ &= \inf_{G \in \mathcal{G}} \int_{\mathbb{T}^2} \left(G + Q^\top e\right) \cdot Q^\top \frac{g(y)}{\sqrt{|g|(y)}} Q \left(G + Q^\top e\right) dy. \end{aligned}$$

However, in two dimensions, for any invertible matrix A we have that

$$Q^\top A^{-1} Q = \frac{A^\top}{\det(A)}, \tag{3.29}$$

so that, since $\det\left(g^{-1}\sqrt{|g|\,(y)}\right) = 1$,

$$e \cdot (Z\,D)^{-1}\,e = \inf_{G\in\mathcal{G}} \int_{\mathbb{T}^2} \left(G + Q^\top e\right) \cdot g^{-1}(y) \left(G + Q^\top e\right)\,\sqrt{|g|\,(y)}\,dy$$
$$= \left(Q^\top e\right) \cdot Z\,D\left(Q^\top e\right),$$

in light of (3.21). Thus

$$\frac{1}{Z}\,e \cdot D^{-1}e = Z\,e \cdot QDQ^\top e = Z\det(D)\,e \cdot D^{-1}\,e,$$

so that $\det(D) = \frac{1}{Z^2}$. □

Remark If D is isotropic, then one can generalise the intuitive argument of Figure 2.1 for the effective diffusion on a 1D surface to the analogous 2D problem. Consider a particle undergoing Brownian motion on a two-dimensional surface which gives no preference to any particular direction, then we expect that the mean first exit time from a fixed circular region on the surface (with surface area Z) would be proportional to the expected first exit time $\mathbb{E}[\tau]$ of a $\mathbb{R}^2$-valued Brownian motion $\sqrt{2}B(t)$ from a circular region with surface Z starting from 0, which is given by $E[\tau] = \frac{Z}{4\pi}$. Since for $\epsilon = n^{-1}$, $n \in \mathbb{N}$ the surface area remains invariant, it follows by taking $\epsilon \to 0$ that the limiting effective diffusion coefficient is proportional to the scalar $\frac{\pi}{Z}$, which is indeed the case.

Proposition 3.4.1 recovers the *area scaling estimate* $D_{as} = \frac{1}{Z}$ to D in the case when D is isotropic. This scaling has been discussed in several previous works, in particular [Halle and Gustafsson, 1997; King, 2004; Gov, 2006; Naji and Brown, 2007]. Although the proof of Proposition 3.4.1 is straightforward, to our knowledge nobody has previously offered a rigorous proof of this result, and Equation (3.27) is also new.

When D is not isotropic, Proposition (3.4.1) still provides us with useful constraints on the anisotropy of the effective diffusion coefficient. Indeed, if λ_1 and λ_2 are the eigenvalues of D with $\lambda_1 \le \lambda_2$, then (3.27) implies that $\lambda_1\lambda_2 = \frac{1}{Z^2}$ and consequently:

$$\frac{1}{Z^2} \le \lambda_1 \le \frac{1}{Z} \le \lambda_2 \le 1. \tag{3.30}$$

We see that if the macroscopic diffusion is unhindered in the direction corresponding to λ_1 then the effective diffusion will be $\frac{1}{Z^2}$ in the orthogonal direction, corresponding to a diffusion on a one-dimensional surface. In the other extreme, if $\lambda_1 = \lambda_2$ then we have an isotropic diffusion coefficient and by the above proposition $\lambda_1 = \lambda_2 = \frac{1}{Z}$.

3.5 A SUFFICIENT CONDITION FOR ISOTROPY

In all of the previous literature regarding lateral diffusion on two-dimensional biological membranes, it is always assumed that the macroscopic diffusion coefficient is isotropic, i.e. a scalar multiple of the identity. While this is a natural assumption, it is clearly not true in general. In this Section we identify a natural sufficient condition to guarantee the isotropy of the effective diffusion coefficient. The condition we will assume is the following:

$$h(x) = h(Qx), \quad x \in \mathbb{R}^2, \tag{3.31}$$

where $Q : \mathbb{R}^2 \to \mathbb{R}^2$ is a $\frac{\pi}{2}$ rotation about some point $\mathbf{O} \in \mathbb{R}^2$. Without loss of generality we assume that $\mathbf{O} = (0, 0)$.

Lemma 3.5.1. *Let $Q \in \mathbb{R}^{2\times 2}$ be any rotation about the origin. If (3.31) holds, then*

$$g^{-1}(Q\,x) = Q\,g^{-1}(x)Q^\top \tag{3.32}$$

and

$$|g|\,(Q\,x) = |g|\,(x), \tag{3.33}$$

for all $x \in \mathbb{R}^2$.

Proof. Applying the chain rule, one can see that $h(Q\,x) = h(x)$ implies

$$(\nabla h)\,(Q\,x) = Q\nabla h(x).$$

Thus, we have that

$$\begin{aligned} g(Q\,x) &= I + Q\,\left(\nabla h(x) \otimes \nabla h(x)\right)\,Q^\top \\ &= Q\,g(x)\,Q^\top, \end{aligned}$$

from which (3.32) and (3.33) follow immediately. □

We now prove that the above condition is a sufficient condition for the effective diffusion coefficient to be isotropic. The proof we present here is based on Schur's lemma [Schur, 1905; James and Liebeck, 2001]. A similar approach can be found in Section 1.5 of [Jikov et al., 1994]. Schur's lemma can be stated as follows

Lemma 3.5.2 (Schur's lemma)**.** *Let $S \in \mathbb{R}^{2\times 2}$ be a symmetric matrix, and let $Q \in \mathbb{R}^{2\times 2}$ be an orthogonal matrix, such that $Q \neq \pm I$. If Q commutes with S, then S is a scalar times the identity.*

Theorem 3.5.3. *If condition (3.31) holds, then D is isotropic.*

Proof. We use a characterisation of D given by (3.21), namely

$$e \cdot De = \frac{1}{Z} \inf_{v \in H^1_{per}(\mathbb{T}^2)} (e + \nabla v(y)) \cdot g^{-1}(y)\,(e + \nabla v(y))\,\sqrt{|g|\,(y)}\,dy.$$

Changing variables $Q\,z = y$, using (3.32) and (3.33),we obtain:

$$
\begin{aligned}
e \cdot De &= \frac{1}{Z} \inf_{v \in H^1_{per}(\mathbb{T}^2)} \int_{\mathbb{T}^2} (e + \nabla v(Q\,z)) \cdot Q\,g^{-1}(z) Q^\top \, (e + \nabla v(Q\,z)) \sqrt{|g|\,(z)}\, dz \\
&= \frac{1}{Z} \inf_{v \in H^1_{per}(\mathbb{T}^2)} \int_{\mathbb{T}^2} \left(Q^\top e + Q^\top \nabla v(Q\,z) \right) \cdot g^{-1}(z) \left(Q^\top e + Q^\top \nabla v(Q\,z) \right) \sqrt{|g|\,(z)}\, dz.
\end{aligned}
$$

Noting that $Q^\top \, \nabla v(Q\,z) = \nabla w(z)$, where $w(z) = v \circ Q(z)$, since Q is a $\frac{\pi}{2}$ rotation, it is clear that $w \in H^1_{per}(\mathbb{T}^2)$ if and only if $v \in H^1_{per}(\mathbb{T}^2)$. It follows that

$$
e \cdot De = e \cdot Q\,D\,Q^\top \, e,
$$

for all $e \in S^1$. Since D is symmetric, it follows by Schur's lemma that D is isotropic.

□

3.6 NUMERICAL METHOD

To compute the effective diffusion coefficient for a general two dimensional surface we use the expression for D given in (3.12). The corrector χ is approximated numerically using a finite element scheme with piecewise linear elements to solve the cell equation (3.9) given in the weak form as

$$
\int_{\mathbb{T}^d} \nabla \chi^{e_i}(y) \cdot g^{-1}(y) \nabla \phi(y) \sqrt{|g|\,(y)}\, dy = - \int_{\mathbb{T}^d} e_i \cdot g^{-1}(y) \nabla \phi(y) \sqrt{|g|\,(y)}\, dy, \quad (3.34)
$$

for all $\phi \in H^1(\mathbb{T}^d)$, where $\{e_i\}_{i=1,\ldots,d}$ is the cartesian basis.

For the approximation of χ we use a regular triangulation of the domain $[0,1]^2$ with mesh-width δ. To impose the periodic boundary conditions of (3.9) we identify the boundary nodes of the mesh periodically. Thus, for $\delta = \frac{1}{M}$, $M \in \mathbb{N}$, the resulting finite element scheme has M^2 degrees of freedom.

The stiffness matrix corresponding to the elliptic differential operator $\mathcal{L}_0$ is assembled using nodal quadrature [Larsson and Thomée, 2009] to compute the local contribution of each triangular element to the stiffness matrix. One can check that this quadrature scheme has order of convergence 2. The load vector corresponding to the right hand side of (3.34) is computed similarly. Thus, the derivatives of the surface map h_{x_1} and h_{x_2} are evaluated only at the nodes of the mesh. For simple surfaces, the derivatives can be computed directly for each node. For more complicated surface maps, the derivatives are computed using a Fourier method [Trefethen, 2000] and then projected onto the mesh nodes using bilinear interpolation. The stiffness matrix S corresponding to $\mathcal{L}_0$ is positive semi-definite, with kernel consisting of constant functions. Since the RHS of the finite element approximation of (3.34) is orthogonal to S, the corresponding matrix equation is solvable.

Once the stiffness matrix and the load vector have been assembled, the resulting symmetric matrix equation is solved using a preconditioned conjugated gradient method [Hestenes and Stiefel, 1952] where a black-box algebraic multigrid method is used as a preconditioner. We used the PETSc library [Balay et al., 1997, 2013a,b] to implement the linear solver. The PETSc library implements routines for the scalable solution of linear and non-linear problems aimed specifically at the numerical approximation of partial differential equations. Apart from transparent parallelization of all the algorithms it implements, it provides a unified interface to a wide family of linear solver algorithms as well as preconditioners, making it possible to change and configure the linear solver and preconditioner directly from the command line. This is extremely beneficial as it allows one to tune the solver and preconditioner experimentally without having to write any additional code. For the examples in this thesis, we used the PETSc implementation of the Preconditioned Conjugate gradient using BoomerAMG, a parallel algebraic multigrid preconditioner [Henson and Yang, 2002].

Given a piecewise linear approximation $\chi_\delta^{e_i}$ of the solution of (3.34) the components of the effective diffusion coefficient are then approximated by

$$e_i \cdot D_\delta \, e_j = \frac{1}{Z_\delta} \int_{\mathbb{T}^d} \left(\nabla \chi_\delta^{e_i}(y) + e_i\right) \cdot \mathcal{P}_\delta \left(g^{-1}\sqrt{|g|}\right)(y) \left(\nabla \chi_\delta^{e_j}(y) + e_j\right) \, dy, \tag{3.35}$$

where $\mathcal{P}_\delta[\cdot]$ is the projection onto the space of piecewise linear functions and

$$Z_\delta = \int_{\mathbb{T}^d} \mathcal{P}_\delta \left(\sqrt{|g|\,(y)}\right) \, dy.$$

To test the order of convergence of this scheme we consider a surface which satisfies the conditions of Theorem 3.5.3, so that the area scaling estimate $D = \frac{1}{Z}\mathbf{I}$ holds. The experimental order of convergence (eoc) for the error $E_\delta := \left|(D_\delta)_{1,1} - \frac{1}{Z}\right|$ is given by

$$eoc = \frac{\log(E_{2\delta}/E_\delta)}{\log(2)}.$$

In Table 3.1 we list the experimental error of convergence for the surface with Monge gauge given by

$$h(x) = \sin(2\pi x_1)\sin(2\pi x_2) \tag{3.36}$$

and we see that the eoc converges to 2 as predicted by theory.

δ	E_h	eoc
$6.250000 \cdot 10^{-2}$	$7.465670 \cdot 10^{-02}$	
$3.125000 \cdot 10^{-2}$	$2.202513 \cdot 10^{-02}$	1.761122
$1.562500 \cdot 10^{-2}$	$5.842453 \cdot 10^{-03}$	1.914504
$7.812500 \cdot 10^{-3}$	$1.485907 \cdot 10^{-03}$	1.975230
$3.906250 \cdot 10^{-3}$	$3.731425 \cdot 10^{-04}$	1.993546
$1.953125 \cdot 10^{-3}$	$9.341470 \cdot 10^{-05}$	1.998005
$9.765620 \cdot 10^{-4}$	$2.335444 \cdot 10^{-05}$	1.999953
$4.882810 \cdot 10^{-4}$	$5.839029 \cdot 10^{-06}$	1.999896

Table 3.1: Table of errors $E_\delta = \left|(D_\delta)_{1,1} - \frac{1}{Z}\right|$ for D_δ for the surface map given in (3.36).

3.7 NUMERICAL EXAMPLES

To illustrate the properties described in the previous sections, we apply the numerical scheme of Section 3.6 to numerically compute the effective diffusion coefficient for diffusions on various classes of surfaces, along with the bounds D^*, D_*, D^+ and the estimate D_{as}.

In Figure 3.1 we consider a surface defined by

$$h(x) = A\sin(2\pi x_1)\sin(2\pi x_2)$$

for varying A. The isotropy condition holds, so that D is isotropic and is given by D_{as}. The Voigt-Reuss bounds are sharp in the weak disorder regime (small A) but become increasingly weak as A increases, with D^* approaching $\frac{1}{2}$ in the strong disorder limit (large A) while D_* converges to 0. We see that for this particular surface D^+ is a relatively tight upper bound to D for varying A. As predicted by Proposition 3.4.1, the area scaling approximation correctly determines the effective diffusion coefficient.

In Figures 3.2 and 3.3 we consider the surface given by

$$h(x) = \sin(2\pi x_1)\sin(6\pi x_2) + A\sin(6\pi x_1)\sin(2\pi x_2).$$

The effective diffusion coefficient will not be isotropic except for $A = 1$. In Figure 3.2 we plot $e_1 \cdot De_1$ for varying A. The Voigt-Ruess bounds D^* and D_* are not tight for any A. The D^+ bound is considerably tighter, and converges to 0 along with D as $A \to \infty$. We also see that area scaling approximation agrees at the point $A = 1$, at which D is isotropic. In Figure 3.3 we plot the maximal and minimal eigenvalues D_{max} and D_{min} of the effective diffusion coefficient. As predicted by (3.30), $\frac{1}{Z}$ lies between D_{max} and D_{min}, meeting at $A = 1$, where they are equal.

In Figure 3.4 we consider a periodic surface given by

$$\begin{aligned} h(x) &= A\exp\left(-\frac{1}{1-|\frac{x-c}{r}|^2}\right), \qquad && |x-c| < r \\ h(x) &= 0 && |x-c| \geq r, \end{aligned} \tag{3.37}$$

that is, h is the standard "bump" function with center $c = \left(\frac{1}{2}, \frac{1}{2}\right)$, radius $r = 0.45$ and amplitude A. The symmetries of the surface fluctuations induce an isotropic effective diffusion, as before. We note that for $A < 1.0$, the effective diffusion is not very sensitive to changes in amplitude, but that it rapidly diminishes if we increase A beyond 2.

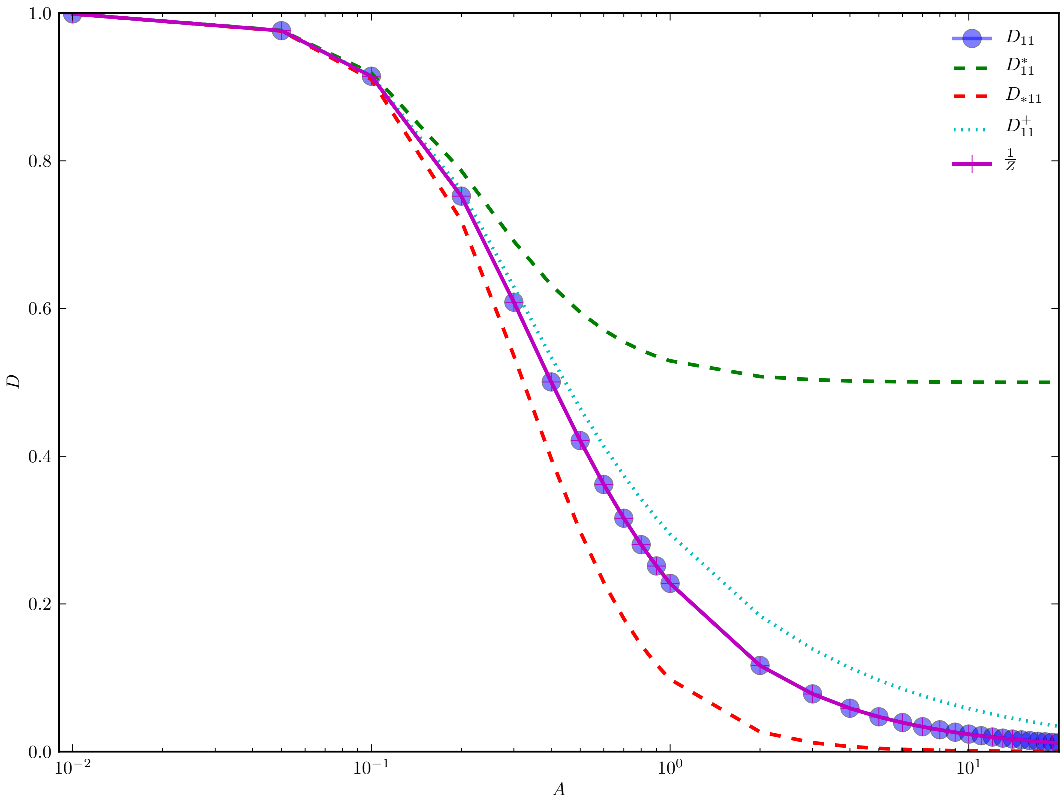


Figure 3.1: The effective diffusion coefficient for an "egg-carton" surface with Monge gauge $h(x) = A\sin(2\pi x_1)\sin(2\pi x_2)$ for varying A. The dots indicate computed values of D. The red line shows D_*, the green line shows D^* and the cyan line denotes the upper bound given by D^+.

3.8 DIFFUSIONS ON SURFACES WITH QUENCHED FLUCTUATIONS

We can apply the results of the previous sections to study the effective behaviour of a particle diffusing laterally on a static surface given by the stationary realisation of the process $\eta(t)$ given in (2.28). This corresponds to the scaling of Case I (i.e. with $(\alpha, \beta) = (1, -\infty)$) of the four scaling regimes described in Section 2.3.3. This particular regime had been previously studied in [Naji and Brown, 2007] for lateral diffusion over a quenched elastic membrane. In this section, we will study how the distribution of the surface realisation affects the averaged effective diffusion. In Section 3.8.1 we focus on the specific case of a fluctuating Helfrich elastic surface.

Each stationary realisation of the surface gives rise to a different effective diffusion coefficient, thus $D(h)$ will be a random variable depending on the particular stationary realisation of the random surface process. In more than one dimension,

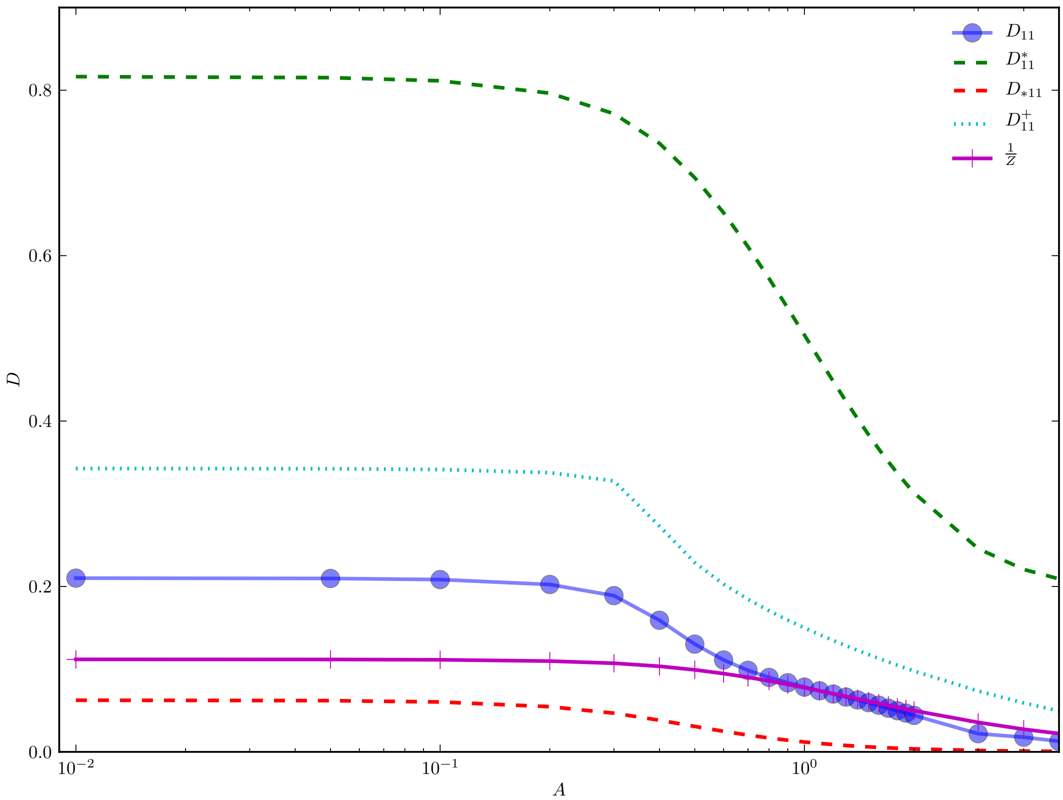


Figure 3.2: The effective diffusion coefficient for $h(x) = \sin(2\pi x_1)\sin(6\pi x_2) + A\sin(6\pi x_1)\sin(2\pi x_2)$. We plot D in the e_1 direction.

not much can be said regarding individual realisations of the surface, beyond the bounds derived in Section 3.3, however, provided we assume a natural generalisation of (3.31), then one can show that the average effective diffusion coefficient is also isotropic.

Consider the stationary measure μ_∞ of the OU process given by (2.31). Let $\mathbb{P}$ be the probability measure on $C(\mathbb{T}^2)$ given by the pushforward of μ_∞ under the map $P : \mathbb{R}^K \to C(\mathbb{T}^2)$ where

$$P(\zeta) = \zeta \cdot e = \sum_{k\in\mathbb{K}} \zeta_k e_k.$$

Denoting by $D(h)$ the effective diffusion coefficient for a particular realisation h of $\mathbb{P}$, we define the average effective diffusion coefficient $\overline{D}$ to be:

$$\overline{D} = \int D(h)\mathbb{P}(dh). \tag{3.38}$$

The first result we show is an analogue of Theorem 3.5.3.

Proposition 3.8.1. *Suppose $d = 2$ and let $Q \in \mathbb{R}^{2\times 2}$ be an orthogonal matrix, such*

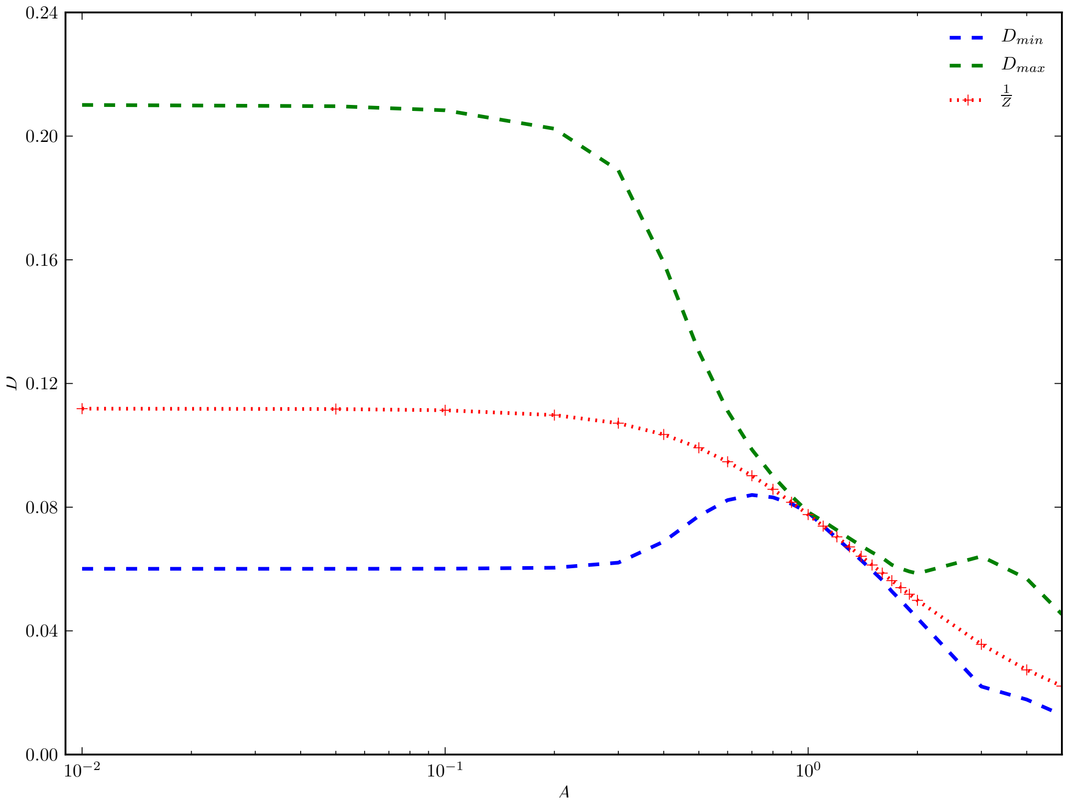


Figure 3.3: The effective diffusion coefficient for $h(x) = \sin(2\pi x_1)\sin(6\pi x_2) + A\sin(6\pi x_1)\sin(2\pi x_2)$. D is anisotropic except for $A = 1$. The maximal and minimal eigenvalues of D, D_{max} and D_{min} respectively, are plotted along with the area scaling approximation $D_{as} = \frac{1}{Z}$, illustrating the bound on the eigenvalues given by (3.30).

that $Q \neq \pm I$. Define $\mathcal{Q} : C(\mathbb{T}^2) \to C(\mathbb{T}^2)$ by

$$(\mathcal{Q}f) = f(Q^\top \cdot).$$

Suppose $\mathbb{P}$ is invariant with respect to $\mathcal{Q}$, that is, $\mathcal{Q}^{-1} \circ \mathbb{P} = \mathbb{P}$, then $\overline{D}$ is isotropic.

Proof. Let h be a realisation of $\mathbb{P}$. Similar to the proof of Theorem 3.5.3, we have that

$$\nabla\left(\mathcal{Q}h\right)(x) = Q\,\nabla h(Q^\top x).$$

Making dependence of g explicit on h, so that $g(x,h) := I + \nabla h(x) \otimes \nabla h(x)$, then

$$g^{-1}(x, \mathcal{Q}h) = Q\, g^{-1}(Q^\top x, h)\, Q^\top$$

and

$$|g|\,(x, \mathcal{Q}h) = |g|\,(Q^\top x, h).$$

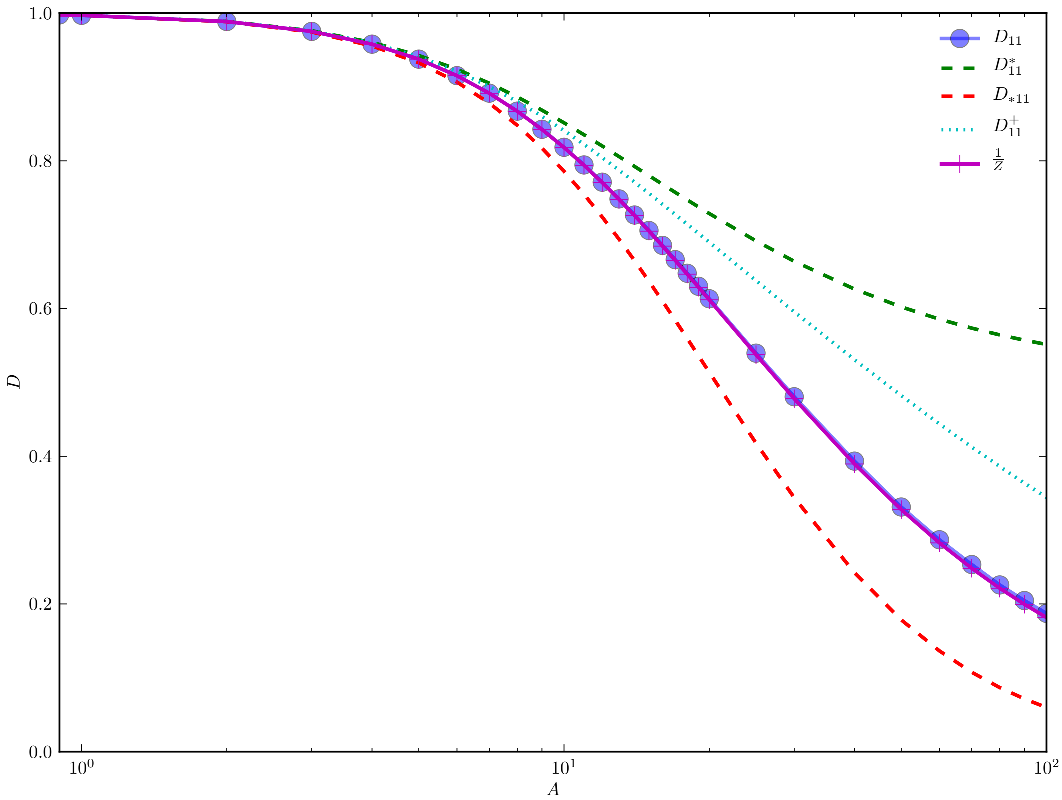


Figure 3.4: The effective diffusion for diffusion on a periodic surface, where each cell is a "bump" with width 0.45 and amplitude A given by the graph of 3.37.

Using a similar argument to that of Proposition 3.5.3 gives

$$
\begin{aligned}
&e \cdot D(\mathcal{Q}h)e \\
&= \inf_{v \in H^1_{per}(\mathbb{T}^2)} \int_{\mathbb{T}^2} (\nabla v(x) + e) \cdot Q\, g^{-1}(Q^\top x, h) Q^\top \left(\nabla v(x) + e\right) \sqrt{|g|\left(Q^\top x, h\right)}\, dy \\
&= \inf_{v \in H^1_{per}(\mathbb{T}^2)} \int_{\mathbb{T}^2} \Big[\Big(Q^\top \nabla v(QQ^\top x) + Q^\top e \Big) \\
&\qquad\qquad \cdot g^{-1}(Q^\top x, h) \Big(Q^\top \nabla v(QQ^\top x) + Q^\top e \Big) \sqrt{|g|\left(Q^\top x, h\right)} \Big]\, dy.
\end{aligned}
$$

Noting that $\nabla\left(v \circ Q\right)(x) = (Q^\top \nabla v)(Qx)$ and that $w = v \circ Q \in H^1_{per}(\mathbb{T}^2)$ iff $v \in$

$H^1_{per}(\mathbb{T}^2)$ it follows that

$$
\begin{aligned}
&e{\cdot}D(\mathcal{Q}h)e \\
&\quad= \inf_{w\in H^1_{per}(\mathbb{T}^2)} \int_{\mathbb{T}^2} \Big[\Big(\nabla w(Q^\top x) + Q^\top e\Big) \\
&\qquad\qquad \cdot g^{-1}(Q^\top x, h)\Big(\nabla w(Q^\top x) + Q^\top e\Big)\sqrt{|g|\,(Q^\top x\, h)}\Big]\,dy \\
&\quad= e\cdot Q\,D(h)Q^\top e.
\end{aligned}
$$

The result follows immediately from the previous relation since using the invariance of the measure $\mathbb{P}$ with respect to $\mathcal{Q}$:

$$
\begin{aligned}
\overline{D} &= \int D(h)\mathbb{P}(dh) \\
&= \int D(\mathcal{Q}h)\mathbb{P}(dh) \\
&= \int QD(h)Q^\top\mathbb{P}(dh) \\
&= Q\,\overline{D}\,Q^\top.
\end{aligned}
\tag{3.39}
$$

It follows from Schur's lemma that $\overline{D}$ is isotropic. $\square$

Although $\overline{D} = \mathbb{E}_P\left[D(h)\right]$ is isotropic, one cannot directly apply the area scaling approximation from Section 3.5 to obtain a closed-form expression for D. Two estimates were proposed for $\overline{D}$ in [Naji and Brown, 2007], namely the averaged area scaling estimate $\overline{D_{as}} = \mathbb{E}_{\mathbb{P}}\left[\frac{1}{Z(h)}\right]$ and the effective medium approximation $\overline{D_{ema}} = \mathbb{E}_{\mathbb{P}}\left[\frac{Z(h)}{\int |g|(y,h)\,dy}\right]$. Based on numerical experiments, the authors conclude that the area scaling estimate D_{as} gives the best agreement with D.

Using a regular perturbation argument, we provide a formal proof of the following result, which shows that, in the low disorder limit when $\delta^2 = \mathbb{E}_{\mathbb{P}}\left|\nabla h\right|^2 \ll 1$, the averaged diffusion coefficient $\overline{D}$ is well approximated by D_{as}, provided the random surface field measure $\mathbb{P}$ is invariant under a rotation operator $\mathcal{Q}$:

Theorem 3.8.2. *Let Q be a $90°$ rotation and define $\mathcal{Q} : C(\mathbb{T}^d) \to C(\mathbb{T}^d)$ to be*

$$\mathcal{Q}\,(h) = h(Q\cdot).$$

Suppose that

$$\mathbb{P}\circ\mathcal{Q}^{-1} = \mathbb{P},$$

and that $\delta^2 = \mathbb{E}_{\mathbb{P}}\left[|\nabla h(y)|^2\right] \ll 1$, then for any unit vector $e\in\mathbb{R}^2$,

$$e\cdot\overline{D}e = \overline{D_{as}} + O(\delta^4), \tag{3.40}$$

where $\overline{D_{as}} = \mathbb{E}_{\mathbb{P}}\left[\frac{1}{Z(h)}\right]$.

Proof. We look for solutions of the cell equation

$$\nabla \cdot \left(\sqrt{|g|\,(y,h)}g^{-1}(y,h)\,(\nabla\chi^e(y,h)+e)\right) = 0. \tag{3.41}$$

satisfying $\int_{\mathbb{T}^2} \chi^e(y,h)\,dy = 0$ and in the form of a power series in δ^2

$$\chi^e(y,h) = \chi_0^e(y,h) + \delta^2\chi_2^e(y,h) + O(\delta^4). \tag{3.42}$$

Write $\nabla h(y) = \delta\nabla h_0(y)$, where $\mathbb{E}[|\nabla h_0|^2] = 1$. By Taylor's theorem we know that

$$\sqrt{1+\delta^2\,|\nabla h_0(y)|^2} = 1 + \frac{\delta^2\,|\nabla h_0(y)|^2}{2} + O(\delta^4). \tag{3.43}$$

Similarly we can write

$$\sqrt{|g|\,(y,h)}g^{-1}(y,h) = \left(1 - \frac{\delta^2\,|\nabla h_0(y)|^2}{2}\right)\left(I + \delta^2 H(y,h_0)\right) + O(\delta^4), \tag{3.44}$$

where $H(y,h_0) = (\nabla h_0)^\perp \otimes (\nabla h_0)^\perp$. Substituting (3.43) and (3.44) into (3.41) neglecting terms of order δ^4 and smaller, it follows that

$$\begin{aligned} - \quad \nabla \quad \cdot \left(\left(1 - \frac{\delta^2\,|\nabla h_0(y)|^2}{2}\right)\left(I+\delta^2 H(y,h_0)\right)\left(\nabla\chi_0^e + \delta^2\nabla\chi_2^e\right)\right) = \\ \nabla\cdot\left(\left(1 - \frac{\delta^2\,|\nabla h_0(y)|^2}{2}\right)\left(I+\delta^2 H(y,h_0)\right)e\right)\end{aligned}$$

Collecting terms of equal orders we get:

$$-\Delta\chi_0^e(y,h_0) = \nabla\cdot e = 0, \tag{3.45}$$

which implies that $\chi_0^e = 0$ as expected. Similarly, collecting $O(\delta^2)$ terms:

$$-\Delta\chi_2^e = \nabla\cdot\left(H(y,h_0)e - \frac{|\nabla h_0(y)|^2}{2}e\right). \tag{3.46}$$

Since the integral of the right hand side is 0, by the Fredholm alternative there is a unique solution $\chi_{e,2}$ with mean zero. Using Property 5 of Proposition 3.3.1, the effective diffusion coefficient D can be computed from χ_e as follows:

$$\begin{aligned} e\cdot D(h)e =& \frac{1}{Z(h)}\int_{\mathbb{T}^2} e\cdot g^{-1}(y,h)e\sqrt{|g|\,(y,h)}\,dy \\ &- \frac{1}{Z(h)}\int_{\mathbb{T}^2}\nabla\chi^e(y,h)\cdot g^{-1}(y,h)\nabla\chi^e(y,h)\sqrt{|g|\,(y,h)}\,dy.\end{aligned} \tag{3.47}$$

Substituting the above expansions in (3.47)

$$
\begin{aligned}
e \cdot D(h) e = &\frac{1}{Z(h)} \int_{\mathbb{T}^2} e \cdot \left(1 - \frac{\delta^2 \left|\nabla h_0(y)\right|^2}{2} + O(\delta^4)\right) \left(I + \delta^2 H(y, h_0)\right) e \, dy \\
&- \frac{\delta^4}{Z(h)} \int_{\mathbb{T}^2} \nabla \chi_2^e(y, h_0) \cdot \left(1 - \frac{\delta^2 \left|\nabla h_0(y)\right|^2}{2} + O(\delta^4)\right) \left(I + \delta^2 H(y, h_0)\right) \nabla \chi_2^e(y) \, dy.
\end{aligned}
$$

Collecting terms of equal powers of δ^2:

$$
e \cdot D(h) e = \frac{1}{Z(h)} + \frac{\delta^2}{Z(h)} \int_{\mathbb{T}^2} e \cdot \left(H(y, h_0) - \frac{\left|\nabla h_0(y)\right|^2}{2}\right) e \, dy + O(\delta^4). \tag{3.48}
$$

Taking expectation with respect to $\mathbb{P}$ and applying Fubini's theorem, we see that the δ^2 term is given by

$$
\int_{\mathbb{T}^2} e \cdot \left(\int \frac{1}{Z(h)} \left[H(y, h_0) - \frac{\left|h_0(y)\right|^2}{2} I\right] \mathbb{P}(dh)\right) e \, dy. \tag{3.49}
$$

By the assumption of invariance with respect to $\mathcal{Q}$, it follows that (3.49) equals 0. Therefore, taking expectation on both sides of (3.48) we have that

$$
\overline{D} = \mathbb{E}_{\mathbb{P}}\left[D(h)\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{1}{Z(h)}\right] + O(\delta^4),
$$

so that the result follows. □

3.8.1 DIFFUSION ON A HELFRICH SURFACE IN THE $(\alpha, \beta) = (1, -\infty)$ REGIME

We can apply the results of the previous section to study the macroscopic behaviour of particles diffusing on a two-dimensional quenched Helfrich elastic membrane. To this end, as in Section 2.2 we set $\mathbb{K} = \{k \in \mathbb{Z}^2 \setminus \{(0,0)\} \mid |k| \leq c\}$, and set the Fourier coefficients of $\eta^\epsilon(t)$ to be

$$
\Gamma = \text{diag}\left(\Gamma_k\right)_{k \in \mathbb{K}} \text{ and } \Pi = \text{diag}\left(\Pi_k\right)_{k \in \mathbb{K}},
$$

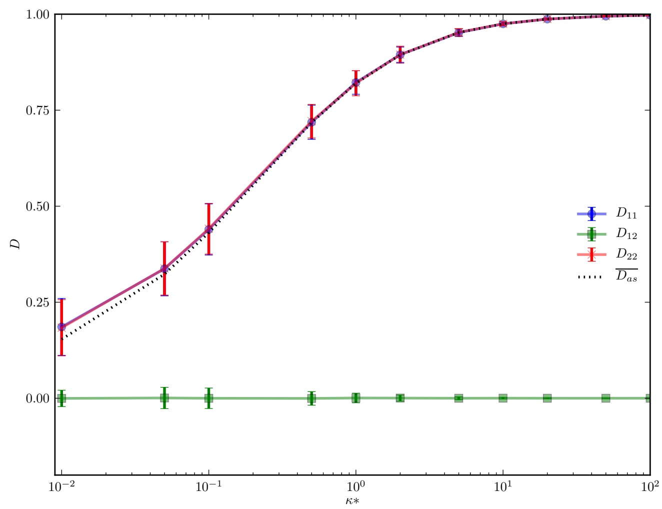
where Γ_k and Π_k are given by (2.15) and (2.16) respectively. The spatial variations are then determined by $\{e_k\}_{k \in \mathbb{K}}$ where $e_k(x) = e^{2\pi i k \cdot x}$.

Since Γ_k and Π_k depend only on $|k|$, the conditions for Proposition 3.8.1 hold trivially, and so the average effective diffusion coefficient is isotropic although individual realisations are not isotropic, with the anisotropy growing as κ^* and σ^* approaching zero. Moreover, for large values of κ^* and σ^*, we expect that the averaged effective diffusion coefficient $\overline{D}$ is well approximated by $\overline{D_{as}}$, by Theorem 3.8.2.

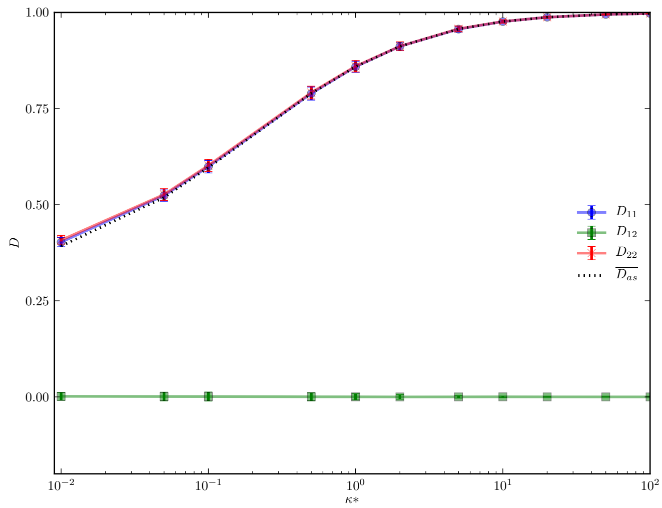
To verify these two predictions we approximate $\overline{D}$ numerically for various parameter values. Realisations of the stationary surface field were generated by sampling the Fourier modes η_k from their respective invariant distribution and performing a Fast Fourier Transform. For each realisation of the surface, $D(h)$ was computed using the numerical scheme described in Section 3.7. In Figure 3.5 we plot $\overline{D}$ for varying bending modulus κ^*, $K = 32$ and for $\sigma^* = 0$ and 100. The effect of κ^* and, to a lesser extent σ^* on the variance of the effective diffusion coefficient is clear. We also plot the averaged area scaling approximation for this case. As predicted by Theorem 3.8.2 for large values of κ^*, which corresponds to the weak disorder limit, that is $\mathbb{E}_{\mathbb{P}}\left[|\nabla h|^2\right] \ll 1$, the averaged area scaling approximation $\overline{D_{as}}$ provides a good approximation to $\overline{D}$, but as $\kappa^* \to 0$ the increasing variance of the fluctuations causes the terms to diverge, with $\overline{D_{as}}$ consistently under-estimating the average diffusion coefficient. This is clear since, by the area scaling approximation applied to $D(h)$:

$$\begin{aligned}\overline{D}_{as} = \mathbb{E}_{\mathbb{P}}\left[\frac{1}{Z(h)}\right] &= \mathbb{E}_{\mathbb{P}}\left[\sqrt{D_{11}(h)D_{22}(h) - (D_{12}(h))^2}\right] \\ &\leq \mathbb{E}_{\mathbb{P}}\left[\sqrt{D_{11}(h)D_{22}(h)}\right] \\ &\leq \sqrt{\mathbb{E}_{\mathbb{P}}\left[D_{11}(h)\right]}\sqrt{\mathbb{E}_{\mathbb{P}}\left[D_{22}(h)\right]} \\ &= \overline{D}.\end{aligned}$$

In Figure 3.6 we plot the distribution of D_{11} along with $\overline{D_{as}}$ and the bounds D_* and D^* for small κ^*, i.e. $\kappa^* \in [10^{-3}, 1]$. The disparity between $\overline{D}$ and $\overline{D_{as}}$ is more apparent.



(a) Plot of the distributions of the components of D for a quenched realisation of a Helfrich surface, for varying κ^* and parameters $K = 32$ and $\sigma^* = 0$.



(b) Similar plot with $\sigma^* = 100$.

Figure 3.5: Plot of the distributions of the components of D for a quenched realisation of a Helfrich surface. Dots denote the mean of the distribution for each κ^* while error bars denote the standard deviation. The dotted line denotes the average area scaling approximation. We note that the diagonal components D_{11} and D_{22} are in very close agreement.

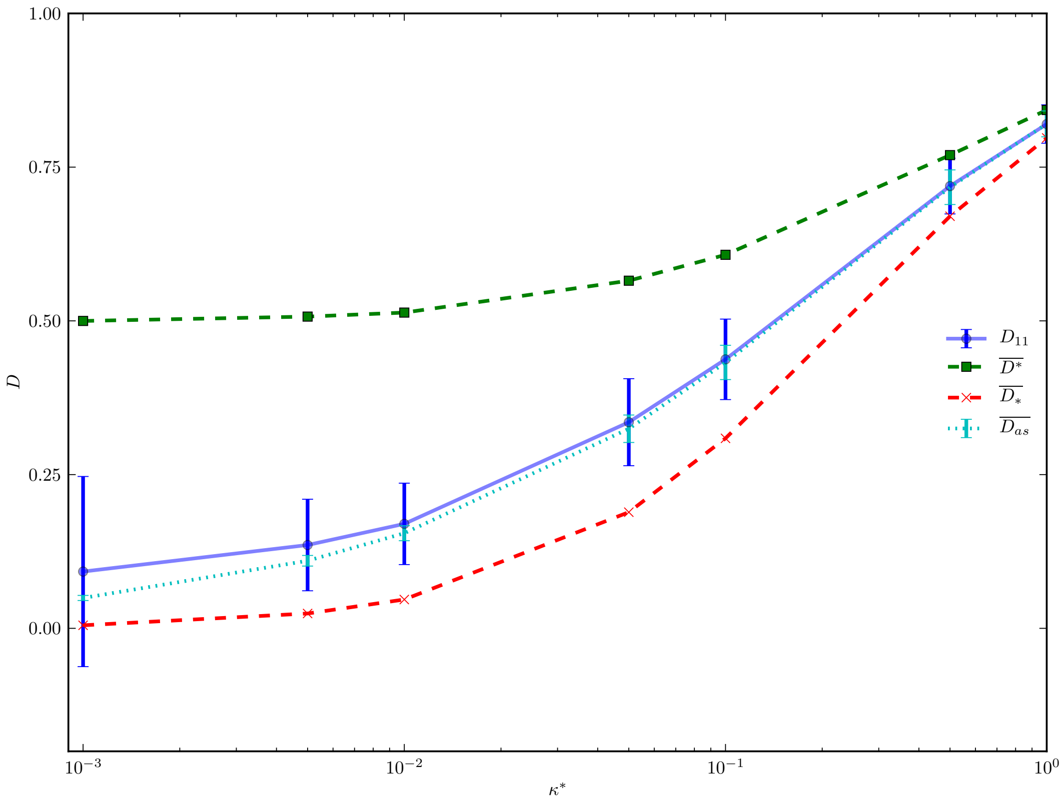


Figure 3.6: Plot of the distributions of D_{11}, $\overline{D_{as}}$, D_* and D^* for the Helfrich elastic membrane model in the Case I regime, with parameters $K = 32$, $\sigma^* = 0$.

3.9 CONCLUSIONS AND FURTHER WORK

The problem of lateral diffusion on a quasi-planar surface with rapid, periodic spatial fluctuations is studied in this chapter. Using formal multiscale expansions we have shown that the large scale behaviour of the lateral diffusion is well-approximated by a Brownian motion on the plane, with constant diffusion coefficient D, which is expressed in terms of the solution of an auxiliary Poisson problem. For $d \geq 2$, D will not have an explicit expression, in general. However, we have been able to derive various properties of D, in particular, bounds on the eigenvalues. When $d = 2$, using a duality transformation result we have proved that if D is isotropic, then the area scaling approximation holds, that is $D = D_{as} = \frac{1}{Z}$, where Z is the average excess surface area. Moreover, we have provided a natural symmetry condition on the surface fluctuation which is sufficient to ensure that D is isotropic.

We have applied these results to the Case I ($(\alpha, \beta) = (1, -\infty)$) scaling regime of (2.28), corresponding to lateral diffusion on a surface given by the stationary realisation of the random field. Since no averaging occurs with respect to the surface fluctuations, the effective diffusion coefficient will depend on the particular realisation of the surface. Instead, we have considered $\overline{D}$, the effective diffusion coefficient averaged over all the stationary surface realisations and identified a sufficient condition for $\overline{D}$ to be isotropic. Moreover, we have shown that, in the weak disorder regime, $\overline{D}$ is well approximated by $\overline{D}_{as}$, as confirmed by numerical experiments. Finally, we have applied these results to the particular model of lateral diffusion on a thermally-excited Helfrich membrane.

There are several natural generalisations one could consider of the simple model that we have studied in this chapter. So far, we have considered surfaces which consist of high-frequency, low-amplitude periodic oscillations about the flat plane in $\mathbb{R}^d$. One generalisation is to consider a surface S^ϵ which consists of a "slowly varying" surface S perturbed along the normal at every point by a rapidly oscillating distance. A second generaliation would be to consider regular parametrized surfaces instead of solely graphs. Although this still precludes the consideration of general closed surfaces, it would permit us to consider surfaces with far more complex geometries than quasi-planar surfaces. A similar problem was considered in [Abdulle and Schwab, 2005] for the stationary heat equation on such a parametrized surface. However, they addressed the problem numerically, formulating a heterogeneous multiscale finite element method for solving the fine-scale problem directly.

For a surface parametrised by a single chart over $\mathcal{O} \subset \mathbb{R}^d$, it is relatively straightforward to formulate this problem as a locally-periodic homogenization problem. Using formal multiscale expansions, one can show that the homogenized equation would be a parabolic PDE defined on $\mathcal{O}$. Moreover, using a duality-transformation result, one can show that the effective diffusion equation would be of the form

$$\frac{\partial u^0(x,t)}{\partial t} = \frac{1}{\sqrt{g_0(x)}}\nabla \cdot \left(\sqrt{g_0(x)}{g_0}^{-1}(x)\nabla u^0(x,t)\right) \quad \text{for } x \in \mathcal{O},$$

where $g_0(x)$ is a symmetric, positive definite matrix. This suggests that $g_0(x)$ can be interpreted as an "effective metric tensor" and the limiting equation will be a diffusion on the "slow manifold" S with metric tensor $g_0(x)$, which is distinct from the induced metric on S. Studying how the rapid fluctuations of the surface affect g_0 would be of interest. In particular, it would be interesting to show whether the homogenized diffusion is in any sense slower than the corresponding diffusion on the slow surface, as one would expect intuitively.

Taking this further, one could consider the problem of diffusion on a closed manifold with rapid-periodic fluctuations. It is not immediately clear how to formulate periodic fluctuations on a surface parametrised by multiple charts. One approach is proposed in [Neuss et al., 2006] who introduce the idea of a locally ϵ-periodic function to study the effective behaviour of a Poisson problem in a domain with a rapidly-oscillating, curved boundary. A similar approach could be adopted here, and the main problem would be to show that the homogenized diffusion equation is independent of the particular chart representation.

Chapter 4

DIFFUSION ON A STATIC SURFACE WITH ERGODIC FLUCTUATIONS

In this chapter we study the long-time behaviour of lateral diffusion on a random surface possessing stationary, ergodic fluctuations. More precisely we wish to study the asymptotic behavior as $\epsilon \to 0$ of the solution of the following SDE on $\mathbb{R}^d$:

$$\begin{aligned} dX^\epsilon(t) =& \frac{1}{\epsilon} \frac{1}{\sqrt{|g|\left(X^\epsilon(t)/\epsilon, h\right)}} \nabla_x \cdot \left(\sqrt{|g|\left(X^\epsilon(t)/\epsilon, h\right)} g^{-1}(X^\epsilon(t)/\epsilon, h) \right) dt \\ &+ \sqrt{2g^{-1}(X^\epsilon(t)/\epsilon, h)}\, dB(t), \end{aligned} \tag{S2}$$

where $g(x,h) = I + \nabla h(x) \otimes \nabla h(x)$ and $|g|\,(x,h)$ denotes the determinant of g for a given realisation of the random field $h(x)$. The backward Kolmogorov equation corresponding to (S2) for a observable $u^\epsilon(x,t)$ is given by

$$\begin{aligned} \frac{\partial u^\epsilon(x,t)}{\partial t} &= \mathcal{L}^\epsilon u^\epsilon(x,t), \qquad && (x,t) \in \mathbb{R}^d \times (0,T], \\ u^\epsilon(x,t) &= u(x), && (x,t) \in \mathbb{R}^d \times \{0\}. \end{aligned} \tag{P2}$$

where

$$\mathcal{L}^\epsilon f(x) = \frac{1}{\sqrt{|g|\,(x/\epsilon, h)}} \nabla_x \cdot \left(\sqrt{|g|\,(x/\epsilon, h)}) g^{-1}(x/\epsilon, h) \nabla_x f(x) \right), \tag{4.1}$$

and where $u \in C_b(\mathbb{R}^d)$.

Our objective is to show that as $\epsilon \to 0$, the process $X^\epsilon(t)$ behaves like a Brownian motion with a constant effective diffusion coefficient D independent of the particular realisation of h. Equivalently, we show that u^ϵ converges point-wise to the solution u^0 of the PDE

$$\begin{aligned}\frac{\partial u^0(x,t)}{\partial t} &= D : \nabla_x \nabla_x u^0(x,t), \qquad (x,t) \in \mathbb{R}^d \times (0,T],\\ u^0(x,t) &= v(x), \qquad\qquad\qquad\quad (x,t) \in \mathbb{R}^d \times \{0\}.\end{aligned} \tag{4.2}$$

This model is a natural generalisation of that of the model considered in the previous chapter and provides a convenient means of studying diffusion on surfaces containing inhomogeneities and micro-structure which are not well represented by periodic functions, making the approach especially attractive for biological applications. Despite this, to our knowledge, the study of lateral diffusion on random surfaces with spatially ergodic fluctuations has not been considered previously, either analytically or numerically.

The problem of homogenization of SDEs and PDEs with random coefficients which are stationary and ergodic has been widely studied since [Papanicolaou et al., 1979] and [Jikov et al., 1994] and is considered to be a fundamental problem in the study of random media. In particular, the problem of homogenization of a diffusion process in a random potential is classical (see for example [De Masi et al., 1989]). While almost all the proofs in this chapter are standard results (see [Komorowski et al., 2012] for a treatise), there does not seem to be a single result which can be cited to prove the homogenisation theorem. Thus, for the sake of completeness, we provide the proof here. The novelty of this chapter lies in the use of random media to model surface inhomogeneities rather than any particular mathematical result. While this model is a natural generalisation of the periodic case, as we shall see the loss of compactness which occurs when moving from a periodic surface to a random surface introduces complications in the analysis. In particular, the random surface case is not amenable to the formal perturbation expansions considered in the previous chapter. For this reason, in this chapter we will adhere to the probabilistic approach to obtain a homogenization result.

The objective of the chapter is the following:

1. We propose a model for lateral diffusion of particles on quasi-planar surfaces possessing stationary, ergodic fluctuations. We then reinterpret the problem of determining the effective behaviour of the process as a standard stochastic homogenization problem, and use standard results to derive a homogenization theorem.

2. We identify natural generalisations of the properties proved in the periodic case and prove them in this case. Indeed, we find that all of the properties proved in Chapter 3 can be shown to hold in some form for this model.

3. We describe a standard approach to computing the effective diffusion coefficient using a periodic approximation [Owhadi, 2003; Bourgeat and Piatnitski, 2004; Alexanderian et al., 2012]. This allows us to approximate the infinite cell problem, required to compute the effective diffusion coefficient, with a periodic cell problem over a suitably large domain. Thus, using this approach

we can apply the numerical scheme described in Section 3.7 to approximate D.

The chapter is organised as follows: in Section 4.1 we describe the model we are considering and identify a set of assumptions on the random field which are sufficient to obtain a homogenization result. In Section 4.2 we introduce the environment process which will play a key role in obtaining a homogenization result. In Section 4.3 we describe the proof of the homogenization theorem for this model, based on the approach of De Masi et al. [1989]. In Section 4.4 we consider the effective diffusion coefficient D. Analogous to the periodic case, we are able to express D as the minimum value of a particular quadratic minimisation problem and use this to obtain upper and lower variational bounds on D. In Section 4.5, using a duality transform argument similar to [Kohler and Papanicolaou, 1982] we relate the determinant of D to the asymptotic excess surface area, and in the case where $d = 2$ and D is isotropic, we obtain a closed-form expression for D. In Section 4.6, by once again applying Schur's lemma, we identify a natural sufficient condition on the law of the random field under which D be isotropic.

In Section 4.7 we describe the periodization method for approximating the effective diffusion coefficent. Using this scheme we provide two numerical examples. The first example is that of a random surface generated by randomly placed protrusions, whose position is determined by a Poisson point process. This model falls under the framework discussed in the previous sections, and we demonstrate that the area scaling approximation holds for this example. The second example we consider is that of a random field generated by a Gaussian random field. Due to the unboundedness of the fluctuations this example will not fall under the above theory, however, we will show that homogenization does appear to occur, at least numerically, and that the area scaling approximation holds all the same.

4.1 PROBLEM FORMULATION AND SET-UP

In this section we introduce the framework required for describing lateral diffusion on a quasi-planar surface perturbed by stationary, ergodic fluctuations. The approach described here is a direct application of the results in Chapter 9 of [Komorowski et al., 2012], and we will follow their approach very closely.
Let Ω be the space of all C^3 functions from $\mathbb{R}^d$ to $\mathbb{R}$ equipped with the Fréchet metric generated by seminorms of the form

$$\|f\|_N = \sup_{|x|\leq N} \sum_{\alpha\leq 3} |\nabla^\alpha f(x)|\,, \qquad N \in \mathbb{N}.$$

Equipped with this metric, one can show that Ω is a Polish space.

For $x \in \mathbb{R}^d$ define the translation operator $\tau_x : \Omega \to \Omega$ by

$$\tau_x(h) = h(\cdot + x), \qquad h \in \Omega.$$

Let $\mathbb{P}$ be a Borel probability measure on the measurable space $(\Omega, \mathcal{B}(\Omega))$ and define the group of translations $\{\tau_x : x \in \mathbb{R}^d\}$. We assume that the following conditions hold:

A. $\mathbb{P}\left(\tau_x^{-1}(B)\right) = \mathbb{P}(B)$, for all $B \in \mathcal{B}(\Omega)$ and $x \in \mathbb{R}^d$. (Stationarity)

B. For any $B \in \mathcal{B}(\Omega)$, $\tau_x(B) = B$ for all $x \in \mathbb{R}^d$ implies that $\mathbb{P}(B) = 0$ or $\mathbb{P}(B) = 1$, (Ergodicity)

C. For all $\delta > 0$, $y \in \mathbb{R}^d$, $\lim_{x\to 0} \mathbb{P}\left[|\tau_x h(y) - h(y)| > \delta\right] = 0$. (Stochastic Continuity)

Moreover, we shall make the following assumption regarding the derivatives of realisations of h:

D. There exists a constant $K > 0$ such that for $\mathbb{P}$-almost surely every realisation of $h \in \Omega$,

$$|\nabla h(x)| + |\nabla\nabla h(x)| \leq K, \qquad x \in \mathbb{R}^d, \quad \mathbb{P} - \text{a.s.} \tag{4.3}$$

Assumption **D** is a very restrictive assumption which precludes considering Gaussian random fields, however without this assumption one encounters insurmountable technical problems when attempting to obtain a homogenization result.

We first define the derivatives with respect to the translation group $\{\tau_x\}$, which are necessary for the formulation of the environment process. For $i \in 1, \dots, d$, let D_i be the $L^2(\mathbb{P})$ generator of τ_x in the e_i direction, that is

$$D_i V := \frac{d}{d\lambda} V(\tau_{(\lambda e_i)} h)\big|_{\lambda=0},$$

in the $L^2(\mathbb{P})$ sense. Assumption **C** permits us to apply Corollary 1.1.6 of [Ethier and Kurtz, 2009], to show that the $\mathcal{D}(D_i)$ are dense in $L^2(\mathbb{P})$. Note that D_i is antisymmetric with respect to the $L^2(\mathbb{P})$ inner product, so that for all $U, V \in \mathcal{D}(D_i) \subset L^2(\mathbb{P})$,

$$\langle D_i U, V\rangle_{L^2(\mathbb{P})} = -\langle U, D_i V\rangle_{L^2(\mathbb{P})}.$$

For $V \in H^1 := \bigcap_{i=1}^d \mathcal{D}(D_i)$, we can then define the gradient to be

$$\mathbb{D}V := (D_i V)_{i=1}^d. \tag{4.4}$$

For a vector field $\mathbf{V} = (V_i)_{i=1}^d$ such that $V_i \in H^1$ we define the divergence to be

$$\mathbb{D} \cdot \mathbf{V} := \sum_{i=1}^d D_i V_i. \tag{4.5}$$

We will model the surface fluctuations by rapidly varying realisations of $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$.

More specifically, analogous to the periodic case, we shall consider a particle diffusing laterally on the rapidly-fluctuating surface

$$S^{\epsilon}(h) = \left\{ \left(x, \epsilon h\left(\frac{x}{\epsilon}\right) \right) \, | x \in \mathbb{R}^d \right\},$$

for a given realisation $h \in \Omega$. In local coordinates we can express the metric tensor as $g\left(\frac{x}{\epsilon}, h\right)$, where $g(x,h)$ is the stationary $\mathbb{R}^{d\times d}$-valued random field given by

$$g(x,h) = I + \nabla h(x) \otimes \nabla h(x).$$

Denote by X_h^{ϵ} the projected trajectory of a particle undergoing lateral diffusion on the surface S_h^{ϵ}. Then $X_h^{\epsilon}(t)$ is the solution of the following Itô SDE

$$dX_h^{\epsilon}(t) = \frac{1}{\epsilon} F(X_h^{\epsilon}(t)/\epsilon, h)dt + \sqrt{2\Sigma(X_h^{\epsilon}(t)/\epsilon, h)}\, dB(t), \tag{4.6}$$

where

$$F(x,h) = \frac{1}{\sqrt{|g(x,h)|}} \nabla_x \cdot \left(\sqrt{|g|\,(x,h)} g^{-1}(x,h) \right) \tag{4.7}$$

and

$$\Sigma(x,h) = g^{-1}(x,h). \tag{4.8}$$

We wish to study the limiting behaviour of $X_h^{\epsilon}(t)$ as $\epsilon \to 0$. Equivalently one can consider the limiting behaviour of the solution $u^{\epsilon}(t,x,h)$ of the backward Kolmogorov equation corresponding to (4.6) which is given by

$$\begin{aligned} \frac{\partial u^{\epsilon}(t,x,h)}{\partial t} &= \frac{1}{\sqrt{|g|\,(x/\epsilon,h)}} \nabla \cdot \left(\sqrt{|g|\,(x/\epsilon,h)} g^{-1}(x/\epsilon,h) \nabla u^{\epsilon}(t,x,h) \right), \quad (t,x) \in [0,T] \times \mathbb{R}^d, \\ u^{\epsilon}(0,x,h) &= v(x), \qquad x \in \mathbb{R}^d, \end{aligned} \tag{4.9}$$

for some $v \in C_b(\mathbb{R}^d)$ independent of h and ϵ.

The case of diffusion on surfaces possessing static, periodic fluctuations, as considered in Chapter 3 can be expressed in the framework described above. Indeed, if $h_0 \in C^3(\mathbb{T}^d)$ extended to $\mathbb{R}^d$ by periodicity, then we can define a random field by

$$h(x) = h_0(x + \zeta),$$

where ζ is distributed according to the Lebesgue measure on $\mathbb{T}^d$. The corresponding probability measure $\mathbb{P}$ on Ω clearly satisfies the conditions **A**-**D**, and moreover the SDE (4.6) and the PDE (4.9) reduce to their periodic counterparts given by (S1) and (P1) respectively for the periodic surface map h_0.

Since Brownian motion is invariant under the diffusive scaling $t \to t/\epsilon^2$, $x \to x/\epsilon$

we can express the process $X_h^\epsilon(t)$ as

$$X_h^\epsilon(t) = \epsilon X_h\left(\frac{t}{\epsilon^2}\right),$$

where $X_h(t)$ is the solution of the Itô SDE

$$dX_h(t) = F(X_h(t), h)dt + \sqrt{2\Sigma(X_h(t), h)}\, dB(t), \tag{4.10}$$

where $B(t)$ is a standard $\mathbb{R}^d$-valued Brownian motion.

For a fixed $h \in \Omega$ the infinitesimal generator of $X_h(t)$ is given by

$$\mathcal{L}_h f = \frac{1}{\sqrt{|g|\,(x,h)}}\nabla_x \cdot \left(\sqrt{|g|\,(x,h)}g^{-1}(x,h)\nabla_x f(x)\right), \qquad f \in C_b^2(\mathbb{R}^d) \tag{4.11}$$

First, we establish the well-posedness of the SDE for $X_h(t)$:

Lemma 4.1.1. *Let X_0 be a random variable with finite second moments, independent of $B(\cdot)$. Then, under assumption (4.3), for $\mathbb{P}$-almost every $h \in \Omega$, the SDE (4.10) has a unique strong solution $X_h(t)$ satisfying $X_h(0) = X_0$. Moreover, the $X_h(t)$ is a Markov diffusion process and possesses a strictly positive continuous transition density $p(t, x, y, h)$.*

Proof. Since realisations of the random field $h(x)$ are almost surely smooth, the coefficients of (4.10) are locally Lipschitz. As in Proposition 2.3.1 we note that

$$\left|g^{-1}(x,h)\right|_2 \leq 1,$$

and that

$$|F(x,h)| \leq C\,|\nabla_x\nabla_x h(x,h)|_2\,,$$

for some constant $C > 0$. Thus, by Assumption (4.3), for $\mathbb{P}$ almost every realisation of $h \in \Omega$ the coefficients of the SDE are bounded. The existence and uniqueness of a strong solution $X_h(t)$ then follows from Theorems 2.2, 3.6 and 4.3 of [Friedman, 2006]. Moreover, since the diffusion coefficient is non-degenerate and the coefficients are bounded, then by Theorem 6.4.6 of [Friedman, 2006] there exists a continuous transition probability $p(t, x \mid x_0, h)$ which solves the Fokker Planck equation for $X_h(t)$. $\square$

4.2 THE ENVIRONMENT PROCESS

The assumption that the random field h is stationary and ergodic with respect to spatial translations is essential to obtaining a limiting diffusion process in the limit as $\epsilon \to 0$. To obtain such a homogenization limit, we need to express the SDE (4.10) in terms of a stationary ergodic Markov process. Following the work of [Kipnis and Varadhan, 1986; De Masi et al., 1989; Papanicolaou et al., 1979], we considered the so-called *environment viewed from the particle*.

We must first express the coefficients of the SDE as stationary random variables on Ω. Defining the random variable $g(h)$ by

$$g(h) := g(0,h) = I + \nabla h(x) \otimes \nabla h(x)\big|_{x=0},$$

we can express the coefficients of the SDE (4.10) as random variables on Ω. Indeed, by defining

$$\begin{aligned} F(h) := F(0,h) &= \frac{1}{\sqrt{|g|\,(x,h)}} \nabla \cdot \left(\sqrt{|g|\,(x,h)}\, g^{-1}(x,h) \right) \Bigg|_{x=0} \\ &= \frac{1}{\sqrt{|g|\,(h)}}\, \mathbb{D} \cdot \left(\sqrt{|g|\,(h)}\, g^{-1}(h) \right), \end{aligned}$$

and,

$$\Sigma(h) := \Sigma(0,h) = g^{-1}(h),$$

we can then express (4.10) as

$$dX_h(t) = F(\tau_{X_h(t)} h)\, dt + \sqrt{2\Sigma(\tau_{X_h(t)} h)}\, dB(t).$$

Define the stochastic process $\zeta_h(t)$ by

$$\zeta_h(t) = \begin{cases} \tau_{X_h(t)} h, & \text{if } t > 0 \\ h & \text{if } t = 0. \end{cases}$$

This stationary, Ω-valued stochatic process known as the *environment viewed from the particle*, and was considered in works such as [Kipnis and Varadhan, 1986], [De Masi et al., 1989] and [Papanicolaou et al., 1979]. It describes the evolution of the environment h which is observed from a frame of reference fixed on the particle. The process $\zeta_h(t)$ is Markovian, with an invariant measure π absolutely continuous with respect to $\mathbb{P}$. In particular, $\zeta_h(t)$ is ergodic, and since $X_h(t)$ is reversible, $\zeta_h(t)$ will also be reversible. The particle trajectory X_h is in some sense driven by $\zeta_h(t)$, indeed we can express $X_h(t)$ in terms of the environment process as follows

$$X_h(t) = X_h(0) + \int_0^t F(\zeta_h(s))ds + \int_0^t \sqrt{2\Sigma(\zeta_h(s))}\, dB(s).$$

In the following lemma we state in detail the basic properties of $\zeta_h(t)$ which were described in the above paragraph.

Lemma 4.2.1 ([Komorowski et al., 2012], Proposition 9.7)**.** *The environment process $\zeta_h(t)$ is Markovian and its transition semigroup $P(t)$ can be written as*

$$P(t)f(h) = \int_{\mathbb{R}^d} p(t, x \,|\, 0, h) f(x,h)\, dx, \qquad f \in L^\infty(\mathbb{P}) \tag{4.12}$$

which can be extended to a positive preserving contraction semigroup on $L^p(\Omega)$ for any

$p \geq 1$. *In particular*

$$\|P(t)f\|_{L^p(\pi)} \leq \|f\|_{L^p(\pi)}, \qquad f \in L^2(\pi).$$

Proof. By the uniqueness and continuity of the transition density, it is straightforward to see that for $t > 0$ and $x, y, z \in \mathbb{R}^d$,

$$p_{\tau_z h}(t, x \mid y, h) = p_h(t, x + z \mid y + z, h). \tag{4.13}$$

Let $n \in \mathbb{N}$, $t > 0$ and $0 \leq t_1 < t_2 < \ldots < t_n = t$. Let f and $f_1, \ldots f_n$ be bounded functions on Ω. Consider for $\delta > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i=1}^n f_i(\zeta_h(t_i)) f(\zeta_h(t+\delta))\right] \\
=&\mathbb{E}\left[\prod_{i=1}^n f_i(X_h(t), h) f(X_h(t+\delta), h)\right].
\end{aligned}$$

Using the Markovianity of $X_h(t)$ and by the Chapman-Kolmogorov theorem, the last term is equal to

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i=1}^n f_i(X_h(t), h) \left(\int_{\mathbb{R}^d} p(\delta, y \mid X_h(t), h) f(y, h)\, dy\right)\right] \\
=&\mathbb{E}\left[\prod_{i=1}^n f_i(X_h(t), h) \left(\int_{\mathbb{R}^d} p(\delta, y - X_h(t) \mid 0, \zeta_h(t)) f(y, h)\, dy\right)\right] \\
=&\mathbb{E}\left[\prod_{i=1}^n f_i(X_h(t), h) \left(\int_{\mathbb{R}^d} p(\delta, y \mid 0, \zeta_h(t)) f(y, \zeta_h(t))\, dy\right)\right] \\
=&\mathbb{E}\left[\prod_{i=1}^n f_i(\zeta_h(t)) \left(P(t) f(\zeta_h(t))\right)\right],
\end{aligned}$$

thus showing that $\zeta_h(t)$ is Markovian and that the semigroup of $\zeta_h(t)$ is indeed given by (4.12). □

By Assumption (4.3) it follows that

$$Z = \int_\Omega \sqrt{|g|\,(h)}\, \mathbb{P}(dh) = \int_\Omega \sqrt{1 + |\nabla h(0)|^2}\, \mathbb{P}(dh) < \infty. \tag{4.14}$$

Define π to be the probability measure on h given by

$$\pi(dh) = \frac{\sqrt{|g|\,(h)}}{Z} \mathbb{P}(dh)$$

Proposition 4.2.2 (Proposition 9.8, [Komorowski et al., 2012]). *The invariant mea-*

sure π is a reversible, ergodic measure of $\zeta_h(t)$. Moreover $C_b^2(\Omega)$ is a core for the L^2-generator $\mathcal{L}$ of $P(t)$ and $\mathcal{L}$ is the unique self-adjoint extension of

$$\hat{\mathcal{L}}f = \frac{1}{\sqrt{|g|\,(h)}}\mathbb{D}\cdot\left(\sqrt{|g|\,(h)}g^{-1}(h)\mathbb{D}f\right), \qquad \mathcal{D}(\hat{L}) = C_b^2(\Omega). \tag{4.15}$$

Finally, we can express the Dirichlet form corresponding to $\mathcal{L}$ as follows

$$\langle(-\mathcal{L})f, f\rangle_{L^2(\pi)} = \frac{1}{Z}\int_\Omega \mathbb{D}f(h)\cdot g^{-1}(h)\mathbb{D}f(h)\sqrt{|g|\,(h)}\mathbb{P}(dh). \tag{4.16}$$

Proof. Since the coefficients of the SDE (4.10) are bounded continuous functions, we can apply the Nash-Aronson type estimates given in Theorem 6.4.5 of [Friedman, 2006] to see that for $\alpha \leq 2$ there exist constants C_α such that

$$|\nabla^\alpha p(t,x\,|\,y,h)| \leq \frac{C_\alpha}{t^{(d+|\alpha|)/2}}\exp\left(-\frac{|x-y|^2}{C_\alpha t}\right).$$

From these estimates, and the expression (4.12) for $P(t)$, it is straightforward to see that $P(C_b^2(\Omega)) \subset C_b^2(\Omega)$. Thus, since $C_b^2(\Omega)$ is dense in $L^2(\pi)$ by [Ethier and Kurtz, 2009, Proposition 3.3] is a core for $\mathcal{L}$. The operator $\hat{\mathcal{L}}$ satisfies

$$\int_\Omega \hat{\mathcal{L}}f(h)\,\pi(dh) = 0, \qquad f \in C_b^2(\Omega),$$

and is symmetric with respect to π. Since $\mathcal{L}$ agrees $\hat{\mathcal{L}}$ on $C_b^2(\Omega)$, it follows that $\mathcal{L}$ is the unique self-adjoint extension of $\left(\hat{\mathcal{L}}, C_b^2(\Omega)\right)$. Thus $\zeta_h(t)$ is reversible with respect to the invariant measure π. It remains to show that π is ergodic. To this end, let $A \in \mathcal{B}(\Omega)$ be a $P(t)$ invariant set, that is, $P(t)\mathbf{1}_A = \mathbf{1}_A$, for $t \geq 0$. Then

$$0 = \langle \mathbf{1}_{A^c}, \mathbf{1}_A\rangle_{L^2(\pi)} = \int_{\mathbb{R}^d}\langle \mathbf{1}_{A^c}(h)p(t,y\,|\,0,h)\mathbf{1}_A(\tau_y h)\rangle\,dy.$$

Since $p(t,0\,|\,y,h) > 0$ for all y, it follows that for all $h \in A^c$, $\tau_y h \in A^c$ and thus it follows by the ergodicity of $\mathbb{P}$ with respect to $\{\tau_y\}_{y\in\mathbb{R}^d}$ that $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$ and so $\pi(A) = 0$ or 1. □

4.3 HOMOGENIZATION RESULT

We now introduce the spaces $\mathcal{H}_1$ and its dual $\mathcal{H}_{-1}$ as defined in [Kipnis and Varadhan, 1986] and [De Masi et al., 1989]. Let $\mathcal{H}_1$ be the completion of the space

$$\left\{\phi \in C_b^2(\Omega)\;\middle|\;\int_\Omega \phi(h)\pi(dh) = 0 \text{ and } \|\phi\|_1 := \langle(-\mathcal{L})\phi,\phi\rangle_{L^2(\pi)} < \infty\right\},$$

with respect to $\|\cdot\|_1$. The dual space $\mathcal{H}_{-1}$ is the completion of the space

$$\left\{\phi \in C_b^2(\Omega) \,\Big|\, \int_\Omega \phi(h)\pi(dh) = 0 \text{ and } \|\phi\|_{-1} < \infty\right\},$$

where the dual norm is given by

$$\|\phi\|_{-1}^2 = \frac{1}{2}\langle \phi, (-\mathcal{L})^{-1}\phi\rangle_{L^2(\pi)} = \sup_{\psi \in H_1} \left\{2\langle \phi, \psi\rangle - \langle(-\mathcal{L})\psi, \psi\rangle\right\}.$$

Note that $\phi \in L^2(\pi)$ lies in $\mathcal{H}_{-1}$ if and only if there exists $C > 0$ such that

$$\langle \phi, \psi\rangle_{L^2(\pi)} \leq C \|\psi\|_1 ,$$

for all $\psi \in \mathcal{H}_1$. Moreover, since $\mathcal{L}$ is positive, self-adjoint

$$\mathcal{H}_1 = \mathcal{D}\left((-\mathcal{L})^{\frac{1}{2}}\right), \qquad \text{and} \qquad \mathcal{H}_{-1} = \mathcal{D}\left((-\mathcal{L})^{-\frac{1}{2}}\right).$$

By Assumption (4.3), the matrix $g(h)$ is uniformly elliptic. This implies that

$$K_1\langle \mathbb{D}\phi, \mathbb{D}\phi\rangle_{L^2(\pi)} \leq \langle \phi, -\mathcal{L}\phi\rangle_{L^2(\pi)} \leq K_2\langle \mathbb{D}\phi, \mathbb{D}\phi\rangle_{L^2(\pi)}, \quad \text{for } \phi \in C_b^1(\Omega),$$

for some positive constants K_1 and K_2. This implies that there is an isomorphism between the spaces $\mathcal{H}_1$ and H_1, and thus, given $\phi \in \mathcal{H}_1$ we are justified in defining the gradient $\mathbb{D}\phi \in L^2(\Omega)$.

As in the periodic case, we wish to decompose the singularly perturbed drift term into a martingale and a remainder term which vanishes as $\epsilon \to 0$ and then apply the martingale central limit theorem to show convergence to a limiting diffusion process. However, unlike in the periodic case, due to the lack of a spectral gap (or equivalently of a Poincaré inequality) for $\mathcal{L}$, the cell equation $-\mathcal{L}\chi = F$ will not be well posed. However, since the resolvent set of $\mathcal{L}$ in $L^2(\pi)$ is $(0, \infty)$, for a fixed unit vector $e \in \mathbb{R}^d$ and $\lambda > 0$, we can consider the following resolvent equation for $\chi_\lambda^e \in L^2(\pi)$:

$$(\lambda I - \mathcal{L})\,\chi_\lambda^e = F^e, \tag{4.17}$$

where $F^e = F \cdot e$.

Lemma 4.3.1. *For any unit vector $e \in \mathbb{R}^d$,*

$$F^e \in L^2(\pi) \cap \mathcal{H}_{-1}$$

Proof. To show that $F^e \in L^2(\pi)$, we note that

$$|F(h)^e| = |F(h) \cdot e| \leq C\,|\nabla\nabla h(x)|_2 ,$$

which is bounded almost surely, by Assumption (4.3). To show that $F^e \in \mathcal{H}_{-1}$ we first note that the centering condition holds, so that

$$\int_\Omega F^e(h)\pi(dh) = 0.$$

Let $\psi \in \mathcal{H}_1$, then

$$\begin{aligned}
\langle F^e, \psi\rangle_{L^2(\pi)} &= \frac{1}{Z}\int_\Omega e\cdot g^{-1}(h)\mathbb{D}\psi(h)\sqrt{|g|\,(h)}\mathbb{P}(dh) \\
&\le \left(\frac{1}{Z}\int_\Omega e\cdot g^{-1}(h)e\sqrt{|g|\,(h)}\mathbb{P}(dh)\right)^{\frac{1}{2}}\left(\frac{1}{Z}\int_\Omega \mathbb{D}\psi\cdot g^{-1}(h)\mathbb{D}\psi(h)\sqrt{|g|\,(h)}\mathbb{P}(dh)\right)^{\frac{1}{2}} \\
&\le \left(\frac{1}{Z}\int_\Omega e\cdot g^{-1}(h)e\sqrt{|g|\,(h)}\mathbb{P}(dh)\right)^{\frac{1}{2}}\|\psi\|_{\mathcal{H}_1} \\
&\le \|\psi\|_{\mathcal{H}_1}.
\end{aligned}$$

It follows that $F^e \in \mathcal{H}_{-1}$ with $\|F^e\|_{\mathcal{H}^{-1}} \le 1$. $\square$

The λ-corrector χ^e_λ can be written as

$$\chi^e_\lambda(h) = \int_0^\infty e^{-\lambda t}P(t)F^e(h),\, dt$$

and so by the contractivity of $P(t)$ we have that $\left\|\chi^e_\lambda\right\|_{L^2(\pi)} \le \frac{1}{\lambda}\left\|F^e\right\|_{L^2(\pi)}$. Moreover, taking the inner product of (4.17) with χ^e_λ we have that

$$\lambda\left\|\chi^e_\lambda\right\|^2_{L^2(\pi)} + \left\|\chi^e_\lambda\right\|^2_{\mathcal{H}_1} = \langle F^e, \chi^e_\lambda\rangle \le \left\|F^e\right\|_{\mathcal{H}^{-1}}\left\|\chi^e_\lambda\right\|_{\mathcal{H}^1},$$

so that $\left\|(\lambda I - \mathcal{L})^{-1}F^e\right\|_{\mathcal{H}_1} \le \left\|F^e\right\|_{\mathcal{H}_{-1}}$. Consequently, we can extend the resolvent operator $(\lambda - \mathcal{L})^{-1}$ from $L^2(\pi)$ to a bounded operator from $\mathcal{H}_{-1}$ to $\mathcal{H}_1$. To be able to obtain a central limit theorem one must show that the λ-correctors decay suitably fast in $L^2(\pi)$ as $\lambda \to 0$ and that they converge to an element in $\mathcal{H}_1$. These two results are typically the core of any stochastic homogenization proof regardless of whether the approach taken is stochastic (as in [Komorowski et al., 2012; De Masi et al., 1989]) or analytic (as in [Papanicolaou et al., 1979]).

Lemma 4.3.2 ([Kipnis and Varadhan, 1986; De Masi et al., 1989]). *There exists $\chi^e \in \mathcal{H}_1$ such that*

$$\lim_{\lambda\to 0}\left\|\chi^e_\lambda - \chi^e\right\|_{\mathcal{H}_1} = 0, \tag{4.18}$$

and

$$\lim_{\lambda\to 0}\lambda\langle\chi^e_\lambda, \chi^e_\lambda\rangle_{L^2(\pi)} = 0 \tag{4.19}$$

Proof. Since $-\mathcal{L}$ is a positive, self-adjoint operator in $L^2(\pi)$, we may associate with

F^e the spectral measure ν_{F^e} relative to the spectral decomposition of $-\mathcal{L}$, such that

$$\int_0^\infty \frac{1}{s}\,\nu_{F^e}(ds) = \langle F^e, (-\mathcal{L})^{-1}F^e\rangle = \|F^e\|^2_{H_{-1}} < \infty.$$

To prove (4.19) we note that

$$\begin{aligned}\lambda\,\|\chi^e_\lambda\|^2_{L^2(\pi)} &= \lambda\langle(\lambda I-\mathcal{L})^{-1}F^e, (\lambda I-\mathcal{L})^{-1}F^e\rangle_{L^2(\pi)} \\ &= \lambda\langle F^e, (\lambda I-\mathcal{L})^{-2}\,F^e\rangle_{L^2(\pi)} = \int_0^\infty \frac{\lambda}{(\lambda+s)^2}\,\nu_{F^e}(ds).\end{aligned}$$

By the the dominated convergence theorem, the integral on the RHS converges to 0 as $\lambda\to 0$, proving (4.19).

Since $F^e \in \mathcal{H}_{-1}$, then $\chi^e := (-\mathcal{L})^{-1}F^e \in \mathcal{H}_1$, then using the spectral representation we have that

$$\begin{aligned}\|\chi^e_\lambda - \chi^e\|^2_{\mathcal{H}_1} &= \left\langle \left((\lambda I-\mathcal{L})^{-1} - (-\mathcal{L})^{-1}\right)F^e, (-\mathcal{L})\left((\lambda I-\mathcal{L})^{-1} - (-\mathcal{L})^{-1}\right)F^e\right\rangle_{L^2(\pi)} \\ &= \int_0^\infty \left(\frac{1}{s+\lambda} - \frac{1}{s}\right)\left(\frac{s}{s+\lambda} - 1\right)\,\nu_{F^e}(ds).\end{aligned}$$

We note that the integrand is dominated by $\frac{1}{s}$, which is integrable with respect to $\nu_{F^e}(ds)$, so that, applying dominated convergence we have that $\left\|\chi^e_\lambda - \chi^e\right\|_{\mathcal{H}_1} \to 0$. $\square$

We can now state the homogenization theorem for $X^\epsilon_h(t)$. The proof is a straightforward extension of the arguments given in [Kipnis and Varadhan, 1986] or [De Masi et al., 1989]. An equivalent, but far more general, approach can be found in [Komorowski et al., 2012]. As in Chapter 3, we use the convention that $(\mathbb{D}\chi)_{ij} = D_j\chi_i$.

Theorem 4.3.3. *Suppose that the conditions **A-D** hold. Then, the process $X^\epsilon_h(t)$ converges weakly in $C([0,T];\mathbb{R}^d)$ to the process $X^0(t)$ which is the unique solution of the Itô SDE*

$$dX^0(t) = \sqrt{2D}B(t), \tag{4.20}$$

where the effective diffusion coefficient is given by

$$D = \frac{1}{Z}\int_\Omega (I+\mathbb{D}\chi(h))\,g^{-1}(h)\,(I+\mathbb{D}\chi(h))^\top\sqrt{|g|\,(h)}\,\mathbb{P}(dh), \tag{4.21}$$

where $\chi = (\chi^{e_i})_{i=1,\ldots,d}$ is the $\mathcal{H}_1$ limit of $(\chi^{e_i}_\lambda)_{i=1,\ldots,d}$ which exists by Lemma 4.3.2.

Proof. Since $C^2_b(\Omega)$ is a core for $\mathcal{L}$, then for each $\lambda > 0$ we can find a sequence $\{\chi^e_{\lambda,n}\}_{n\in\mathbb{N}}$ such that

$$\chi^e_{\lambda,n} \to \chi^e_\lambda \qquad \text{and} \qquad \mathcal{L}\chi_{\lambda,n} \to \mathcal{L}\chi^e_\lambda,$$

in $L^2(\pi)$ as $n \to \infty$. Applying Itô's formula and taking the limit $n \to \infty$, we have, for a fixed unit vector $e \in \mathbb{R}^d$:

$$\chi_\lambda^e(\zeta_h(t)) = \chi_\lambda^e(\zeta_h(0)) + \lambda \int_0^t \chi_\lambda^e(\zeta_h(s))\, ds - \int_0^t F^e(\zeta_h(s))\, ds + \int_0^t \sqrt{2\Sigma(\zeta_h(s))}\mathbb{D}\chi_\lambda^e(\zeta_h(s))\, dB(s).$$

Substituting in the equation for $X_h(t)$ and writing $\chi_\lambda = (\chi_\lambda^{e_1}, \ldots, \chi_\lambda^{e_d})^\top$, we can express the process $X_h^\epsilon(t)$ as follows

$$X_h^\epsilon(t) = \epsilon X_h(t/\epsilon^2) = \epsilon\left(\chi_\lambda(\zeta_h(0)) - \chi_\lambda(\zeta_h(t/\epsilon^2))\right) - \epsilon\lambda \int_0^{t/\epsilon^2} \chi_\lambda(\zeta_h(s))\, ds + \epsilon \int_0^{t/\epsilon^2} \sqrt{2\Sigma(\zeta_h(s))}\, (I + \mathbb{D}\chi_\lambda(\zeta_h(s)))^\top\, dB(s).$$

Define the $\mathbb{R}^d$-valued martingales $M_\lambda(t)$ and $M(t)$ to be

$$M_\lambda(t) = \int_0^t \sqrt{2\Sigma(\zeta_h(s))}\, (I + \mathbb{D}\chi_\lambda(\zeta_h(s)))^\top\, dB(s) \tag{4.22}$$

and

$$M(t) = \int_0^t \sqrt{2\Sigma(\zeta_h(s))}\, (I + \mathbb{D}\chi(\zeta_h(s)))^\top\, dB(s).$$

Due to the isomorphism between H_1 and $\mathcal{H}_1$, the derivative $\mathbb{D}\chi$ is well-defined and contained in $(L^2(\pi))^d$, thus $M(t)$ is an $L^2(\pi)$-square integrable martingale. Moreover, by the stationarity of $\zeta_h(t)$ we have that

$$\begin{aligned}
&\mathbb{E}_\pi \left|M_\lambda(t) - M(t)\right|^2 \\
&= \int_0^t \left(\int_\Omega (\mathbb{D}\chi_\lambda(\zeta_h(s)) - \mathbb{D}\chi_\lambda(\zeta_h(s)))g^{-1}(h)(\mathbb{D}\chi_\lambda(\zeta_h(s)) - \mathbb{D}\chi_\lambda(\zeta_h(s)))^\top \pi(dh)\right) ds \\
&= t\left\|\chi_\lambda - \chi\right\|_{\mathcal{H}_1}^2,
\end{aligned} \tag{4.23}$$

which converges to zero as $\lambda \to 0$. Define the $\mathbb{R}^d$-valued process $R_\lambda(t)$ to be

$$R_\lambda(t) = (M_\lambda(t) - M(t)) + (\chi_\lambda(\zeta_h(s)) - \chi_\lambda(\zeta_h(t))) + \lambda \int_0^t \chi_\lambda(\zeta_h(s))\, ds \tag{4.24}$$

Now set $\lambda = \epsilon^2$ and define $R^\epsilon(t) := \epsilon R_{\epsilon^2}(t/\epsilon^2)$ and $M^\epsilon(t) = \epsilon M(t/\epsilon^2)$ so that

$$\epsilon X_h(t/\epsilon^2) = M^\epsilon(t) + R^\epsilon(t).$$

The quadratic variation $[\![M^\epsilon]\!](t)$ of the martingale $M^\epsilon(t)$ is given by

$$[\![M^\epsilon]\!](t) = \epsilon^2 \int_0^{t/\epsilon^2} \left(I + \mathbb{D}\chi(\zeta_h(s))\right) g^{-1}(\zeta_h(s)) \left(I + \mathbb{D}\chi(\zeta_h(s))\right)^\top \, ds, \tag{4.25}$$

which by the ergodic theorem (Theorem 1.5.6, [Krengel and Brunel, 1985]), converges in L^1 to the deterministic function $2D\,t$ where D is given by (4.21). Apply the martingale central limit theorem, as stated in Theorem 7.1.4 of [Ethier and Kurtz, 2009], it follows that as $\epsilon \to 0$, M^ϵ converges weakly in $C([0,T];\mathbb{R}^d)$ to the process $\sqrt{2D}\,B(\cdot)$. It remains to show that $R^\epsilon(t)$ converges weakly to zero. It follows from (4.26) of Lemma 4.3.4 that any finite dimensional distribution of the process $R^\epsilon(t)$ converges to 0. We show that the family of processes is tight by verifying the two conditions of Theorem 8.2 of [Billingsley, 2009]. Condition (i) follows trivially and condition (ii) follows immediately from (4.27) of Lemma 4.3.4. $\square$

Lemma 4.3.4. *The remainder term $R^\epsilon(t) = \epsilon R_{\epsilon^2}(t/\epsilon^2)$, for R_λ given by (4.24) satisfies*

$$\lim_{\epsilon\to 0} \|R^\epsilon(t)\|_{L^2(\pi)} = 0, \tag{4.26}$$

and for each $\delta > 0$,

$$\lim_{\epsilon\to 0} \pi \left[\sup_{0\le t\le T} |R^\epsilon(t)| > \delta \right] = 0. \tag{4.27}$$

Proof. The remainder term $R^\epsilon(t)$ satisfies

$$|R^\epsilon(t)| \le \underbrace{\left|\epsilon M_{\epsilon^2}(t/\epsilon^2) - M^\epsilon(t)\right|}_{\text{A}} + \underbrace{\left|\epsilon\chi_{\epsilon^2}(\zeta_h(0)) - \epsilon\chi_{\epsilon^2}(\zeta_h(t/\epsilon^2))\right|}_{\text{B}} + \underbrace{\epsilon^3 \int_0^t \left|\chi_{\epsilon^2}(\zeta_h(s/\epsilon^2))\right| \, ds}_{\text{C}} \tag{4.28}$$

Equation (4.26) follows immediately from (4.23) and the stationarity of $\zeta_h(t)$. We now prove (4.27). Denote by $[\![Q]\!](t)$ the quadratic variation of the process $Q(t)$. We can apply the Burkholder-Davis-Gundy inequality (see Section IV.4 of [Revuz and Yor, 1999]) to term (A) to see that there exists a constant C such that

$$\mathbb{E}_\pi \left(\sup_{0\le t\le T} \left|\epsilon M_{\epsilon^2}(t/\epsilon^2) - \epsilon M(t/\epsilon^2)\right| \right)^2 \le C\,\epsilon^2 \mathbb{E}_\pi [\![M_{\epsilon^2} - M]\!](T/\epsilon^2) \le CT \, \|\chi_{\epsilon^2} - \chi\|^2_{\mathcal{H}_1}, \tag{4.29}$$

which converges to 0 by (4.18). Term (C) is equally straightforward since

$$\mathbb{E}_\pi \left(\epsilon^3 \sup_{0\le t\le T} \int_0^{t/\epsilon^2} |\chi_{\epsilon^2}(\zeta_h(s))| \, ds \right)^2 ds = \epsilon^6 \int_0^{T/\epsilon^2} \mathbb{E}_\pi \, |\chi_{\epsilon^2}(\zeta_h(s))|^2 \le \epsilon^4 \, \|\chi_{\epsilon^2}\|^2_{L^2(\pi)}, \tag{4.30}$$

which converges to zero by (4.19). Obtaining pointwise bounds for (B) is delicate. We refer the reader to Lemma 2.6 of [De Masi et al., 1989] and the subsequent discussion to see that for fixed $e \in S^{d-1}$ and for all $\delta > 0$

$$\lim_{\epsilon \to 0} \pi \left(\epsilon \sup_{0 \le s \le T} \left| \chi^e_{\epsilon^2}(\zeta_h(s)) \right| \ge \delta \right) = 0. \tag{4.31}$$

Combining (4.29), (4.30) and (4.31) we obtain (4.27) via Markov's inequality. $\square$

Corollary 4.3.5. *Let $u^\epsilon(t, x, h)$ be the solution to the backward Kolmogorov equation (4.9), with initial condition $v \in C_b(\mathbb{R}^d)$, independent of ϵ. Then*

$$\lim_{\epsilon \to 0} \mathbb{E}_{\mathbb{P}} \left| u^\epsilon(t, x, h) - u^0(t, x) \right| = 0, \qquad \textit{for all } (t, x) \in [0, T] \times \mathbb{R}^d, \tag{4.32}$$

where $u^0 : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the solution of

$$\frac{\partial u^0(t, x)}{\partial t} = D : \nabla\nabla u^0(t, x), \qquad (t, x) \in [0, T] \times \mathbb{R}^d, \tag{4.33}$$

where D is given by (4.21).

Proof. Applying the Feynmann-Kac formula we can express the solution $u^\epsilon_h(t, x)$ as

$$u^\epsilon_h(t, x) = \mathbb{E}_\pi v \left(X^\epsilon_{x,h}(t) \right), \tag{4.34}$$

where $X^\epsilon_{x,h}(t)$ is the solution of (4.6) with $X^\epsilon_{x,h}(0) = x$. By the stationarity of the random field, this can be rewritten as

$$\mathbb{E}_\pi v \left(x + X^\epsilon_{\tau_x h}(t) \right) = \mathbb{E}_\pi v \left(x + \epsilon X_{\tau_x h}(t/\epsilon^2) \right), \tag{4.35}$$

which converges to $u^0(t, x) := \mathbb{E}_{\mathbb{Q}} v(x + X^0(t))$, by Theorem 4.3.3. But $u^0(t, x)$ is the solution of equation (4.33) by the Feynman-Kac formula. The convergence in $L^1(\mathbb{P})$ stated in (4.32) follows by the dominated convergence theorem.

$\square$

4.4 PROPERTIES OF THE EFFECTIVE DIFFUSION COEFFICIENT

When $d = 1$, we are able to obtain an analytic expression for the gradient of the corrector, and thus able to obtain a closed-form expression for D.

Proposition 4.4.1. *For $d = 1$ we have that*

$$\mathbb{D}\chi(h) = \frac{\sqrt{|g|\,(h)}}{Z} - 1$$

and the effective diffusion is given by

$$D = \frac{1}{Z^2},$$

where Z is given by (4.14).

Proof. This proof is a direct application of Theorem 9.22 of [Komorowski et al., 2012]. $\square$

The interpretation of the constant Z is less clear than in the periodic case. Since the random field is stationary and ergodic, by the ergodic theorem given in Theorem 2.1 of [Taylor] we can write Z as

$$Z = \int_{\Omega} \sqrt{g(h)}\mathbb{P}(dh) = \lim_{R\to\infty} \frac{1}{R^2} \int_{[0,R]^2} \sqrt{g(x,h)}\, dx, \tag{4.36}$$

for $\mathbb{P}$-almost every realisation h. For fixed R, the term

$$\frac{1}{R^2} \int_{[0,R]^2} \sqrt{g(x,h)}\, dx$$

gives the relative excess surface area generated by the surface with respect to the base plane in the square region $[0, R]^2$. Thus one can interpret Z as the average excess surface area for a particular realisation of the surface. Equation (4.36) also provides one with a means to numerically approximate Z.

In two dimensions or more it is not in general possible to obtain an explicit expression for $\mathbb{D}\chi$. This is compounded by the fact that χ, is obtained as the limit of the λ-correctors χ_λ in the abstract space $\mathcal{H}_1$. However, it is possible to express $\mathbb{D}\chi$ as the unique weak solution of an elliptic problem in $(L^2(h))^d$, which can be used to identify the effective diffusion as the minimal value of a quadratic functional over the space of mean-zero, curl-free vector functions, analogous to the periodic case. Using this variational formulation one can easily obtain bounds on the effective diffusion coefficient. By considering the dual minimisation problem one can also obtain lower bounds for D. The approach taken here follows the exposition given in Chapter 10 of [Komorowski et al., 2012].

Denote by $(L^2(\mathbb{P}))^d$ the space of $\mathbb{R}^d$ valued functions of h with components in $L^2(\mathbb{P})$, equipped with the inner product

$$\langle U, V\rangle = \sum_{i=1}^{d} \langle U_i, V_i\rangle_{\mathbb{P}}.$$

The gradient operator $\mathbb{D}$ defined in (4.4) maps H_1 into $(L^2(\mathbb{P}))^d$. Define $L^2_{pot}(\mathbb{P})$ to be the range of $\mathbb{D}$ in $(L^2(\mathbb{P}))^d$. Let $L^2_c(\mathbb{P})$ be the space of constant vector fields in $(L^2(\mathbb{P}))^d$, that is

$$L^2_c(\mathbb{P}) = \text{span}\{e_i \,|\, i = 1, \dots, d\},$$

where e_i is the i^{th} coordinate basis element of $\mathbb{R}^d$. Finally, define $L^2_{div}(\mathbb{P})$ to be the orthogonal complement of $L^2_c(\mathbb{P}) \oplus L^2_{pot}(\mathbb{P})$ in $(L^2(\mathbb{P}))^d$, so that we obtain the following Helmholtz decomposition

$$(L^2(\mathbb{P}))^d = L^2_{pot}(\mathbb{P}) \oplus L^2_{div}(\mathbb{P}) \oplus L^2_c(\mathbb{P}).$$

The space $L^2_{div}(\mathbb{P}) \oplus L^2_c(\mathbb{P})$ can be interpreted as the space of divergence-free vector fields with square integrable components. The following result shows that $\mathbb{D}\chi$ can be expressed as the unique weak solution of a cell equation posed in $\left(L^2(\mathbb{P})\right)^d$. Note that in the case where the fluctuations are periodic this reduces to the "periodic" cell equation given by (3.9).

Proposition 4.4.2. *For any $e \in \mathbb{R}^d$ such that $|e| = 1$, $V = \mathbb{D}\chi^e$ is the unique solution of the problem*

$$\begin{aligned} V &\in L^2_{pot}(\mathbb{P}), \\ \sqrt{|g|\,(h)}g^{-1}(h)\,(e + V(h)) &\in L^2_{div}(\mathbb{P}). \end{aligned} \tag{4.37}$$

Proof. We first prove (4.37) for the case where $e = e_i$, $i \in 1, \ldots, d$. Clearly $\mathbb{D}\chi^{e_i} \in L^2_{pot}(\mathbb{P})$ and $\mathbb{D}\chi^{e_i}_\lambda \in L^2_{pot}(\mathbb{P})$, by definition. Equation (4.17) can be written in weak form as

$$\lambda\langle \chi^{e_i}_\lambda, \phi\rangle_\pi + \frac{1}{Z}\int_\Omega (e_i + \mathbb{D}\chi^{e_i}_\lambda)\cdot g^{-1}\mathbb{D}\phi(h)\sqrt{|g|\,(h)}\,\mathbb{P}(dh) = 0, \qquad \phi \in C^1_b(\Omega)$$

Since $\mathbb{D}\chi^{e_i}_\lambda \to \mathbb{D}\chi^{e_i}$ in $\left(L^2(\mathbb{P})\right)^d$ and $\lambda\chi^{e_i}_\lambda \to 0$ in $L^2(\mathbb{P})$, taking $\lambda \to 0+$ it follows that

$$\int_\Omega (e_i + \mathbb{D}\chi^{e_i}(h))\cdot g^{-1}(h)V(h)\,\sqrt{|g|\,(h)}\,\mathbb{P}(dh) = 0, \tag{4.38}$$

for all $V \in L^2_{pot}(\mathbb{P})$, so that $\mathbb{D}\chi^{e_i}$ is a solution of (4.37). To show that $\mathbb{D}\chi^e$ solves (4.37) for an arbitrary unit vector $e \in \mathbb{R}^d$ we simply note that $\mathbb{D}\chi^e = \sum_{i=1}^d \langle e, e_i\rangle_{\mathbb{R}^d}\mathbb{D}\chi^{e_i}$.

To show uniqueness, suppose V is another solution and let $\delta V = V - \mathbb{D}\chi^e$. Then substituting δV in equation (4.38) we see that

$$0 = \int_\Omega \delta V(h)\cdot g^{-1}(h)\delta V(h)\sqrt{|g|\,(h)}\mathbb{P}(dh) \geq c_* \left\|\delta V\right\|^2_{(L^2(\mathbb{P}))^d},$$

by the uniform ellipticity of $g^{-1}\sqrt{|g|\,(h)}$. □

Analogously to the corresponding result given in Proposition 3.3.1, D can be expressed as the minimum of a particular quadratic functional. Indeed, if $e \in \mathbb{R}^d$ is a unit vector, then the effective diffusion coefficient in the direction e can be written as

$$e \cdot De = \frac{1}{Z}\inf_{V\in L^2_{pot}(\mathbb{P})}\int_\Omega (e + V(h))\cdot g^{-1}(h)\,(e + V(h))\,\sqrt{|g|\,(h)}\mathbb{P}(dh). \tag{4.39}$$

This can be seen by noting that the weak cell equation (4.37) is the Euler-Lagrange equation (4.39), and that $\mathbb{D}\chi^e$ is the unique minimiser of this variational problem. In particular, by substituting $V = 0$ we obtain a (rough) upper bound for the effective diffusion coefficient.

One can also obtain a lower bound for D simply by extending the domain over which (4.39) to $L^2_{pot}(\mathbb{P}) \oplus L^2_{div}(\mathbb{P})$, that is

$$e \cdot De \geq \frac{1}{Z} \inf_{\substack{V \in (L^2(h))^d, \\ \int V \mathbb{P}(dh) = 0}} \int_\Omega (e + V(h)) \cdot g^{-1}(h) \, (e + V(h)) \sqrt{|g| \, (h)} \mathbb{P}(dh).$$

As noted in Proposition 3.3.1, this minimisation problem can be solved directly to obtain a closed-form expression for the minimum value, giving the following lower bound.

$$e \cdot De \geq e \cdot \frac{1}{Z} \left(\int_\Omega \frac{g(h)}{\sqrt{|g| \, (h)}} \mathbb{P}(dh) \right)^{-1} e.$$

We summarize the above properties of D in the following theorem

Theorem 4.4.3. *Let $e \in \mathbb{R}^d$ be a unit vector, then*

1. *D is a symmetric, positive definite matrix*

2. *D is the minimum value of the following minimisation problem:*

$$e \cdot De = \frac{1}{Z} \inf_{V \in L^2_{pot}(\mathbb{P})} \int_\Omega (e + V(h)) \cdot g^{-1}(h) \, (e + V(h)) \sqrt{|g| \, (h)} \, \mathbb{P}(dh), \quad (4.40)$$

   and χ is the unique minimiser of this functional.

3. *The effective diffusion coefficient D satisfies the following inequality*

$$e \cdot D_* e \leq e \cdot De \leq e \cdot D^* e,$$

   where

$$D^* = \frac{1}{Z} \int_\Omega g^{-1}(h) \sqrt{|g| \, (h)} \mathbb{P}(dh), \quad (4.41)$$

   and

$$D_* = \frac{1}{Z} \left(\int_\Omega \frac{g(h)}{\sqrt{|g| \, (h)}} \mathbb{P}(dh) \right)^{-1}. \quad (4.42)$$

4. *In particular D satisfies*

$$\frac{1}{Z^2} \leq e \cdot De \leq 1. \quad (4.43)$$

Remark The upper bound in (4.43) implies that the macroscopic diffusion coefficient is always less than the microscopic diffusion coefficient (which is rescaled to 1), as expected.

4.5 THE AREA SCALING APPROXIMATION

When $d = 2$, the duality transformation argument applied in the periodic case carries over to the stochastic random field case almost without modification. Thus we obtain a relationship between the determinant of D and the asymptotic excess surface area Z. As before, if the effective diffusion coefficient is additionally isotropic we also recover the analogous result to the area scaling approximation, namely that $D = \frac{1}{Z}\mathbf{I}$.

Proposition 4.5.1. *For $d = 2$, the effective diffusion coefficient satisfies the following relationship*

$$\det(D) = \frac{1}{Z^2}. \tag{4.44}$$

Moreover, if D is isotropic, then D can be written explicitly as

$$D = \frac{1}{Z}\mathbf{I}. \tag{4.45}$$

Proof. We follow the approach used in [Kohler and Papanicolaou, 1982]. We first note that Thompson's duality principle (Section 2.6.2 of [Mei and Vernescu, 2010]) applies equivalently in the space $\left(L^2(\mathbb{P})\right)^d = L^2_{\mathrm{pot}}(\mathbb{P}) \oplus L^2_{\mathrm{div}}(\mathbb{P}) \oplus L^2_{\mathrm{c}}(\mathbb{P})$. Applying this result, we have the following relation

$$e \cdot (ZD)^{-1}\, e = \inf_{F \in L^2_{div}(\mathbb{P})} \int_\Omega (F(h) + e) \cdot \frac{g(h)}{\sqrt{|g|\,(h)}}\, (F(h) + e)\; \mathbb{P}(dh) \tag{4.46}$$

Let $Q : \mathbb{R}^2 \to \mathbb{R}^2$ denote a $\frac{\pi}{2}$ rotation about the origin in $\mathbb{R}^2$. Given $F \in L^2_{div}(\mathbb{P})$, define

$$\mathcal{Q}F(h) = (QF)(h).$$

The map $\mathcal{Q} : \left(L^2(\mathbb{P})\right)^d \to \left(L^2(\mathbb{P})\right)^d$ defined by

$$\mathcal{Q}G(h) = (QG)\,(h),$$

is an isomorphism between the sets

$$\left\{\mathbb{D}f \mid f \in C^1_b(\Omega)\right\} \text{ and } \{F \in (C^1_b(\Omega))^2 \mid \int_\Omega F(h)\mathbb{P}(dh) = 0 \text{ and } \mathbb{D}\cdot F = 0\},$$

which can be extended to an isomorphism between $L^2_{pot}(\mathbb{P})$ and $L^2_{div}(\mathbb{P})$. Thus (4.46) can be rewritten as

$$\begin{aligned}
e \cdot (Z\,D)^{-1}\, e &= \inf_{G \in L^2_{pot}(\mathbb{P})} \int_\Omega (QG + e) \cdot \frac{g(h)}{\sqrt{|g|\,(h)}}\, (QG + e)\; \mathbb{P}(dh) \\
&= \inf_{G \in L^2_{pot}(\mathbb{P})} \int_\Omega \left(G + Q^\top e\right) \cdot Q^\top \frac{g(h)}{\sqrt{|g|\,(h)}} Q \left(G + Q^\top e\right)\; \mathbb{P}(dh).
\end{aligned}$$

However, in two dimensions, for any invertible matrix A we have that

$$Q^\top A^{-1} Q = \frac{A^\top}{\det(A)}, \tag{4.47}$$

so that, since $\det\left(g^{-1}\sqrt{|g|\,(y)}\right) = 1$,

$$\begin{aligned} e\cdot (Z\,D)^{-1}\,e &= \inf_{G\in L^2_{pot}(\mathbb{P})} \int_\Omega \left(G + Q^\top e\right)\cdot g^{-1}(h)\left(G + Q^\top e\right)\,\sqrt{|g|\,(h)}\,\mathbb{P}(dh) \\ &= \left(Q^\top e\right)\cdot Z\,D\left(Q^\top e\right). \end{aligned}$$

Thus

$$\frac{1}{Z}\,e\cdot D^{-1}e = Z\,e\cdot QDQ^\top e = Z\det(D)\,e\cdot D^{-1}\,e,$$

so that $\det(D) = \frac{1}{Z^2}$. □

4.6 A SUFFICIENT CONDITION FOR ISOTROPY

In this section we identify a natural symmetry condition on the random field which is sufficient to guarantee that the effective diffusion coefficient is isotropic. As before, we restrict our interest to the case where $d = 2$. By generalising the approach of Section 3.5, in Theorem 4.6.2 we use Schur's lemma to show that if the law of the random field is invariant under some non-trivial rotation, then the effective diffusion will be isotropic.

Let $Q \in \mathbb{R}^{2\times 2}$ be a proper orthogonal matrix. Define the operator $\mathcal{Q}^\top$ on h to be

$$\mathcal{Q}^\top h(x) = h(Q^\top x) \qquad x\in\mathbb{R}^2.$$

Clearly $\mathcal{Q}^\top$ is an isometry on h which induces the following transformations on the metric tensor.

Lemma 4.6.1. *Let $Q\in\mathbb{R}^{2\times 2}$ be any rotation about the origin, then*

$$g^{-1}(x, \mathcal{Q}^\top h) = Qg^{-1}(Q^\top x, h)Q^\top \tag{4.48}$$

and

$$|g|\,(x, \mathcal{Q}^\top h) = |g|\,(Q^\top x, h), \tag{4.49}$$

for all $x\in\mathcal{D}$.

Proof. It follows from the chain rule that

$$\mathbb{D}\left(\mathcal{Q}^\top h\right)(x) = \nabla h\circ Q^\top(x) = Q\,(\mathbb{D}h)\,(Q^\top x). \tag{4.50}$$

From this, it is clear that

$$
\begin{aligned}
g(x, \mathcal{Q}^\top h) &= I + \mathbb{D}\left(\mathcal{Q}^\top h\right)(x) \otimes \mathbb{D}\left(\mathcal{Q}^\top h)\right)(x) \\
&= I + Q\left[(\mathbb{D}h)(Q^\top x) \otimes (\mathbb{D}h)(Q^\top x)\right] Q^\top \\
&= Q\, g(Q^\top x, h) Q^\top.
\end{aligned}
$$

$\square$

We can now state the sufficient condition for the effective diffusion coefficient to be isotropic.

Theorem 4.6.2. *Let $Q \in \mathbb{R}^{2\times 2}$ be a rotation about some point by an angle not equal to 0 or π. Suppose that the random field measure $\mathbb{P}$ is invariant with respect to the corresponding operator $\mathcal{Q}^\top$, that is*

$$
\mathbb{P} \circ \left(\mathcal{Q}^\top\right)^{-1} = \mathbb{P}.
$$

Then D is isotropic.

Proof. By stationarity, we may assume that Q is a rotation about the origin. The set $\{\mathbb{D}f \mid f \in C^1_b(\Omega)\}$ is dense in $L^2_{pot}(\mathbb{P})$, thus we may minimise (4.39) over this set. Moreover, since $\mathcal{Q}^\top$ is measure-preserving we can make the substitution $h \to \tau_x \mathcal{Q}^\top h$ in (4.39) to get

$$
\begin{aligned}
e \cdot De = \frac{1}{Z} \inf_{f \in C^1_b(\Omega)} \int_\Omega \Big[&\left(\nabla f(x, \mathcal{Q}^\top h) + e\right) \\
&\cdot g^{-1}(x, \mathcal{Q}^\top h)\left(\nabla f(x, \mathcal{Q}^\top h) + e\right)\sqrt{|g|\,(x, \mathcal{Q}^\top h)}\Big] \mathbb{P}(dh),
\end{aligned}
$$

Substituting (4.48) in the above we obtain

$$
\begin{aligned}
e \cdot De = \frac{1}{Z} \inf_{f \in C^1_b(h)} \int_\Omega \Big[&Q^\top\left(Q\nabla f(Q^\top x, h) + e\right) \\
&\cdot g^{-1}(Q^\top x, h) Q^\top \left(Q \nabla f(x, h) + e\right)\sqrt{|g|\,(Q^\top x, h)}\Big] \mathbb{P}(dh).
\end{aligned}
$$

Using the fact that Q is orthogonal and $\mathbb{P}$ is invariant under translations τ_y for any $y \in \mathbb{R}^2$ we obtain

$$
\begin{aligned}
e \cdot De &= \frac{1}{Z} \inf_{f \in C^1_b(\Omega)} \int_\Omega \left(\mathbb{D}f(h) + Q^\top e\right) \cdot g^{-1}(h)\left(\mathbb{D}f(h) + Q^\top e\right)\sqrt{|g|\,(h)}\mathbb{P}(dh) \\
&= \left(Q^\top e\right) \cdot D\left(Q^\top e\right) \\
&= e \cdot \left(QDQ^\top\right) e.
\end{aligned}
$$

Since e is arbitrary, it follows that $D = QDQ^\top$ and so, by applying Schur's lemma (Lemma 3.5.2) it follows that the effective diffusion D is isotropic. $\square$

Remark Unlike in the periodic analogue of this result given in Theorem 3.5.3 we are not restricted to 90° rotational symmetries.

Remark As an immediate corollary of Theorem 4.6.1 we note that it is sufficient that the random field is isotropic, i.e. the two point covariance is of the form $C(x, y) = C(|x-y|)$ for D to be isotropic.

4.7 NUMERICALLY APPROXIMATING THE EFFECTIVE DIFFUSION COEFFICIENT

In the two dimensional case, when it is known a priori that the effective diffusion coefficient D is isotropic, then it is relatively straightforward to approximate D numerically, making use of the closed form expression (4.45), via (4.36). However, when D is not isotropic one must resort to other approaches. Unlike in the periodic case the expression (4.21) for D does not lend itself to numerical approximation, due to the fact that the corrector χ exists only in the abstract space $\mathcal{H}_1$. A commonly used method to numerically approximate D is via aperiodic approximation, as follows

1. For a fixed $R > 0$ define $F_R(x, h)$ and $\Sigma_R(x, h)$ to be the coefficients given by

$$F_R(x, h) = F(\tau_{(x \textbf{ mod } B_R)} h), \text{ and } \Sigma_R(x, h) = \Sigma(\tau_{(x \textbf{ mod } B_R)} h), \tag{4.51}$$

   where $F(\cdot)$ and $\Sigma(\cdot)$ are the drift and diffusion coefficients given by (4.7) and (4.8) respectively and where $B_R = R\,\mathbb{T}^2$.

2. Let $X_R(t)$ be the solution of the Itô SDE

$$X_R(t) = F_R(X_R(t), h)\, dt + \sqrt{2\Sigma_R(X_R(t), h)}\, dB(t),$$

   and consider the corresponding periodic homogenization problem which gives rise to an effective diffusion coefficient $D_R(h)$. Then $D_R(h)$ can be approximated numerically using the finite element scheme of Section 3.6.

By trivially modifying the arguments given in [Owhadi, 2003; Bourgeat and Piatnitski, 2004], as $R \to \infty$, one can show that the periodic approximation $D_R(h)$ will converge to D for $\mathbb{P}$ almost every $h \in \Omega$. Moreover it is possible to obtain algebraic rates of convergence depending on the uniform mixing coefficient of the random field. To illustrate this numerical method and explore the theoretical results detailed in this chapter we consider two examples.

4.7.1 EXAMPLE 1

In the first example we consider the problem of lateral diffusion on a "random protrusion surface" a two-dimensional random surface comprised of randomly distributed protrusions, represented as "bump" functions in the form of (3.37) where the centers of the bumps are determined by a Poisson point process with constant

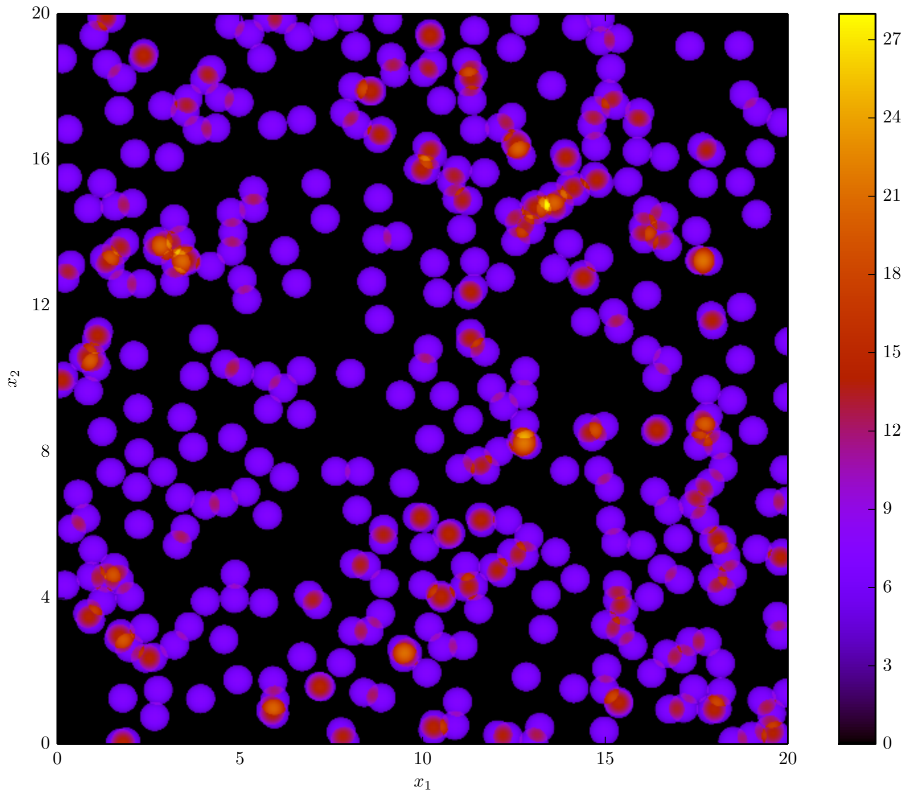


Figure 4.1: A realisation of the poisson generated random field $h(x)$ with homogeneous intensity $\lambda = 1$, over the interval $[0, 20]^2$. Note that the inclusions in the surface are permitted to overlap.

intensity λ. More precisely, we consider a surface which can be formally written as the the graph of

$$h(x) = \sum_i f(x - x_i), \tag{4.52}$$

where $\{x_i\}_{i\in\mathbb{N}}$ is a realisation of a Poisson point process and

$$f(x) = \begin{cases} \alpha \exp\left(-\frac{1}{1-x^2}\right) & |x| < 1 \\ 0 & |x| \geq 1, \end{cases} \tag{4.53}$$

where $\alpha > 0$ is a constant amplitude. This example is a natural generalisation of the example given in Figure 3.4 in Chapter 3. A realisation of this random field over the region $[0, 20]^2$ is plotted in Figure 4.1. We note that the inclusions are allowed to overlap.

Similar models for random media are widely studied, in particular in the study of random Schrödinger operators [Pastur, 1971; Leschke et al., 2005]. A Poisson point process with intensity λ satisfies the following two fundamental properties [Daley and Vere-Jones, 2007]:

1. For every bounded, closed set B, the counting measure

$$N(B) := |\{i : x_i(h) \in B\}|,$$

   is a Poisson process distributed with mean $\lambda\mu(B)$, where $\mu(B)$ is the Lebesgue measure of B.

2. If $B_1, \ldots B_m$ are disjoint regions then $N(B_1), N(B_2), \ldots N(B_M)$ are independent.

The Poisson point process is completely characterised by its Laplace functional, indeed if ϕ is a positive smooth function with compact support on $\mathbb{R}^2$ and we define

$$\nu(\phi) = \sum_i \phi(x_i(h)),$$

then

$$\mathbb{E}\left[e^{-\nu(\phi)}\right] = \exp\left[\lambda \int \left(e^{-\phi(y)} - 1\right) dy\right]. \tag{4.54}$$

From equation (4.54) we see immediately that the Poisson point process is stationary with respect to spatial translations, and thus so is $h(x)$. Furthermore, it is well known that the random field $h(x)$ is ergodic with respect to spatial translations (see Proposition 2.6 of [Meester, 1996]). Realisations of the field $h(x)$ are clearly smooth and bounded with all derivatives bounded, so that this random field satisfies the conditions of Theorem 4.3.3, which guarantees the existence of a homogenization limit. Moreover, it is straightforward to see that since the intensity λ is constant, the conditions of Theorem 4.6.2 holds, and so the effective diffusion coefficient D is isotropic and thus equal to $\frac{1}{Z}$. In the remainder of this section we demonstrate all these results numerically using the numerical scheme detailed at the start of Section 4.7.

Properties 1 and 2 of the Poisson point process can be used to generate realisations of $h(x)$ over the domain $B_R = [0, R]^2$. To sample the centers of the inclusions in this region, we first sample the number of points N from the Poisson distribution with mean value λR^2. The centers of the inclusions $x_1(h), \ldots, x_N(h)$ are sampled uniformly in $[0, R]^2$. The gradient of the random field over B_R is given by

$$\nabla_x h(x, h) = \sum_{i=1}^{N(h)} \nabla_x f(x - x_i).$$

By rescaling the above field from $[0, R]^2$ to $[0, 1]^2$, for a fixed realisation h, we then use the finite element scheme described in Section 3.6 to numerically compute the periodized effective diffusion coefficient $D_R(h)$.

We use this method to generate realisations of $D_R(h)$ for intensity $\lambda = 1$. We use a starting mesh-size of 2^{-6}, stopping when the relative error of $D_R(h)$ between successive refinements is 10^{-2}. In Figure 4.2, we plot, for different values of R, the ergodic average $\frac{1}{N}\sum_{i=1}^{N} D_R(h_i)$, where h_i, $i = 1, \dots N$ are N independent realisations of the random field. For larger values of R the ergodic averages appear to converge faster to the expected value. This however comes at the cost of requiring increasingly smaller mesh-sizes to maintain a constant error of the finite element approximation as R increases.

In Figure 4.3, for each R we plot the mean value of $D_R(h)$. The dashed line denotes the predicted value of D, given by $\frac{1}{Z}$, and we see that there is relatively good agreement between the mean value of $D_R(h)$ and D for large values of R. As predicted by Theorem 4.6.2 the average of $D_R(h)$ will converge to an isotropic diffusion coefficient for large R. In Figure 4.4 we plot the standard deviation of the distribution of each component of $D_R(h)$ for $R \in [1, 15]$, and observe the decay of the variance as R goes to infinity.

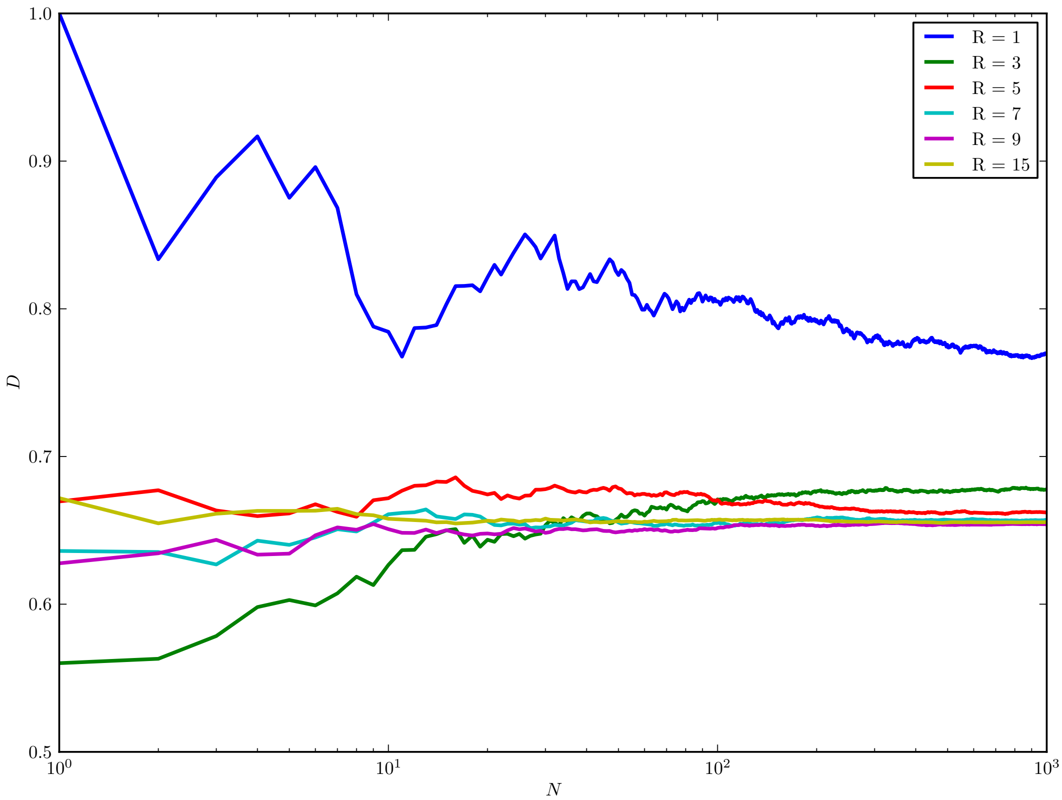


Figure 4.2: Ergodic average of the diffusion coefficient for diffusion on $h(x)$ given by a realisation of the random protrusion surface, plotted for differing values of R and $\alpha = 1$. In each case, as $N \to \infty$ converges to the average $\mathbb{E}\left[e_1 \cdot D_R(h)e_1\right]$.

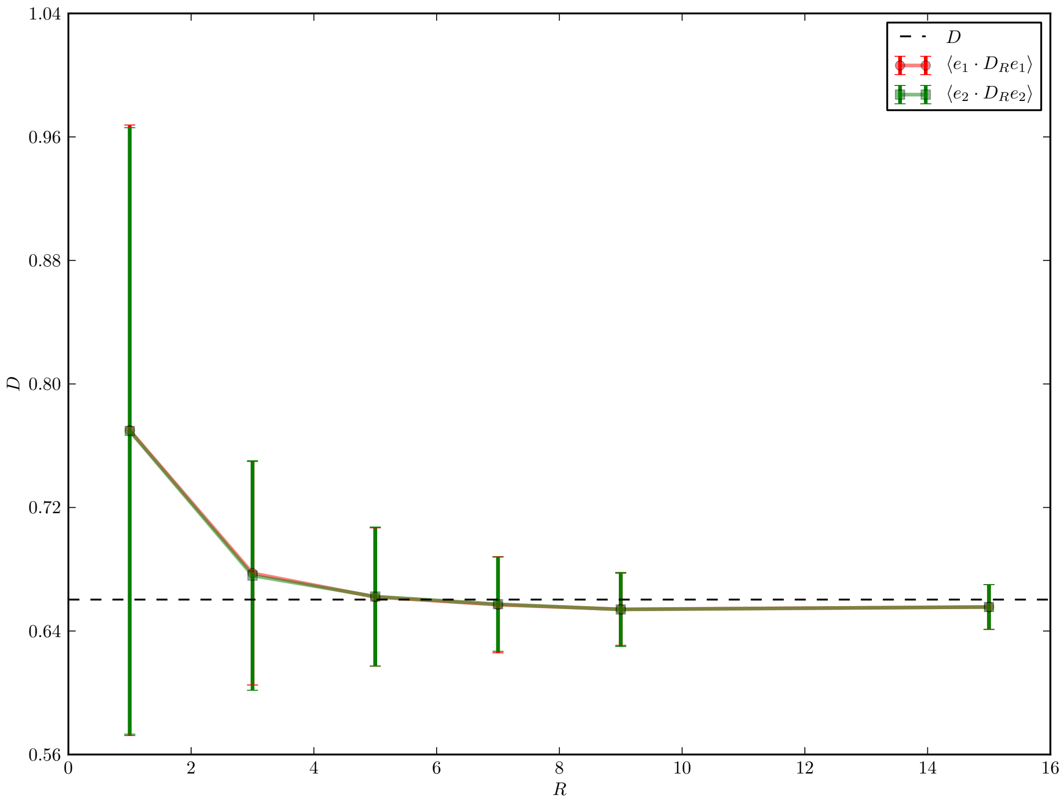
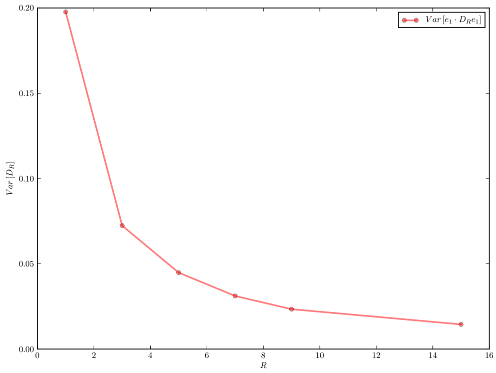
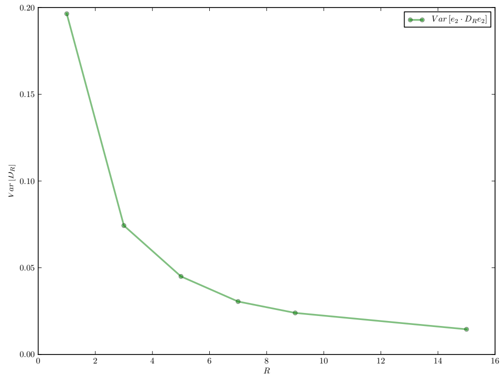


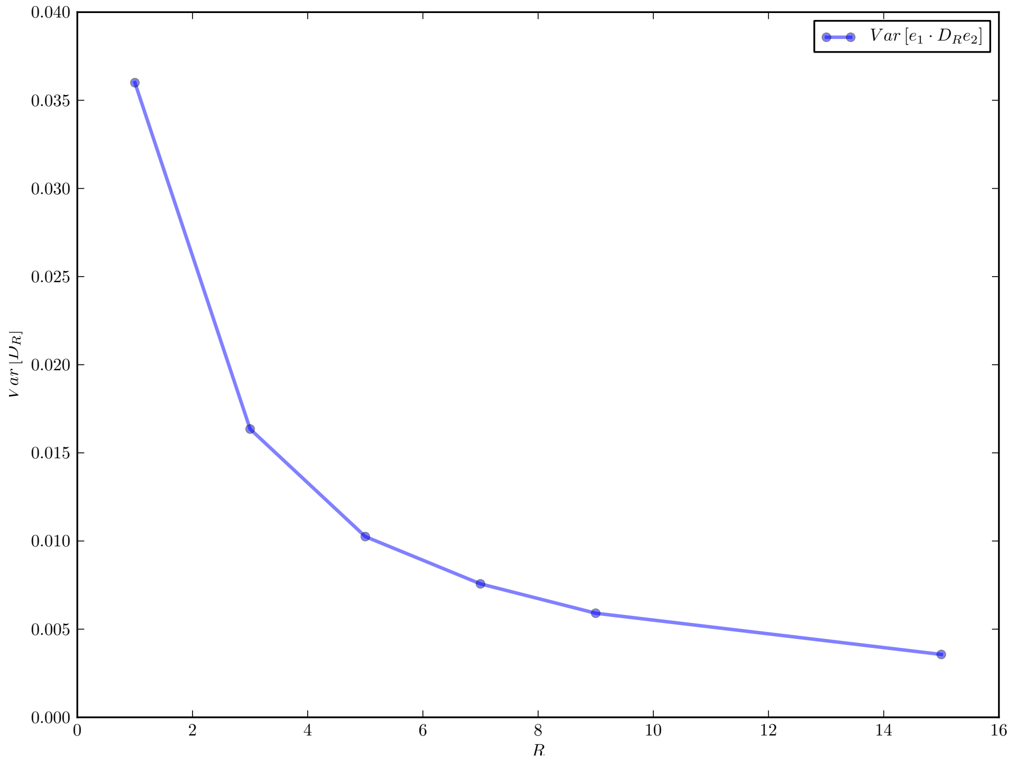
Figure 4.3: Plots of the average of the components $D_R(h)$ for increasing values of R, for the random protrusion surface. For each R, 10^3 realisations of D_R were generated. We note that, as predicted, $D_R(h)$ converges to an isotropic diffusion coefficient. The dashed line indicates the value of $\frac{1}{Z}$ which is the effective diffusion coefficient as predicted by Proposition 4.5.1.



(a) Plots of the standard deviation of $e_1 \cdot D_R(h)e_1$.



(b) Similar plot with $e_2 \cdot D_R(h)e_2$.



(c) Similar plot with $e_1 \cdot D_R(h)e_2$.

Figure 4.4: Plots of the standard deviation of the components of $D_R(h)$ for varying R for the random protrusion surface.

4.7.2 EXAMPLE 2

In the second example we consider a surface generated by a two-dimensional stationary Gaussian random field. Due to the unbounded support of the random field fluctuations, this case does not fall into the framework of this chapter. However we show that homogenization does appear to occur and that the conclusions of Theorems 4.3.3 and 4.6.2 and Proposition 4.5.1 appear to still hold in this case.

We consider an isotropic Gaussian random field $h : \mathbb{R}^2 \to \mathbb{R}$ with mean zero and exponentially decaying autocorrelation given by

$$c_\alpha(r) = e^{-\pi\alpha|r|^2}, \tag{4.55}$$

where α is a positive constant. By Bochner's theorem [Reed and Simon, 1975, Theorem IX.9], $c_\alpha(x-y)$ defines a covariance operator C_α, and a Gaussian measure on $L^2(\mathbb{R}^d)$ with mean 0 and covariance C_α. Realisations of $h(x)$ are almost-surely smooth. To see this, we note that for any $h \in \Omega$ and $x, y \in \mathbb{R}^2$ we have that

$$\mathbb{E}\left|h(x) - h(y)\right|^2 = 2\left(1 - e^{-\pi\alpha|x-y|^2}\right) \leq 2\left|x-y\right|^r,$$

for any $r > 0$. It follows from the Kolmogorov continuity theorem [Stroock, 2011, Theorem 4.3.2] that $h(\cdot)$ is almost-surely Hölder continuous with exponent strictly less than one. Moreover, for $i = 1, 2$ the mean-square derivative $\partial_{x_i} h(x)$ is also Gaussian, with covariance $\partial^2_{x_i, y_i} c(x, y)$, therefore applying the Kolmogorov continuity theorem again we see that $\nabla h(x)$ is Hölder continuous. By extending the argument to higher derivatives it follows that $h(x)$ is smooth.

To be able to simulate a realisation of $h(x)$ over the domain $B_R = [0, R]^2$, we must make the assumption that the field decorrelates over sufficiently long distances. Indeed, we will assume α and L are sufficiently large so that

$$c_\alpha(r) \approx 0 \text{ for } |r| > R. \tag{4.56}$$

The approach we take is as follows. We first generate a realisation of a centered Gaussian random field $u(x)$ over $[-R, R]^2$ with periodic boundary conditions and with autocorrelation

$$c_\alpha^{per}(r) = e^{-\pi\alpha|r|^2_{per}},$$

where $|\cdot|_{per}$ denotes the Euclidean norm on $[-R, R]^2$ induced by the norm on $2R\mathbb{T}^2$. Let $\mathbb{K} = \mathbb{Z}^2 \setminus \{(0, 0)\}$. Provided assumption (4.56) holds then it is straightforward to see that the restriction $u|_{[0,R]^2}(x)$ is a centered Gaussian random field with autocorrelation $c_\alpha^{per}(r) \approx c_\alpha(r)$, so that $u|_{[0,R]^2}$ can be used to generate an approximate sample of $h(x)$ over this region.
By rescaling the domain to $\mathbb{T}^2$ the random field $u(x)$ has the following Karhunen-

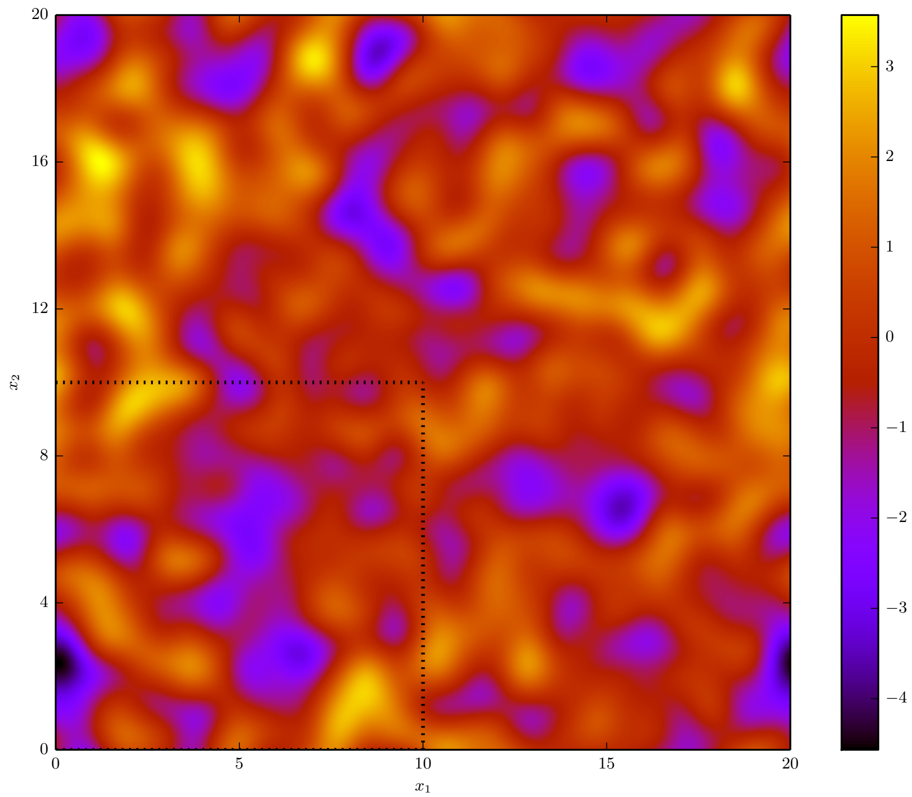


Figure 4.5: A realisation of the Gaussian random field $u(x)$ generated by the algorithm described in Section 4.7.2, with $\alpha = 1$, $M = 1024$ and $R = 10$ (Note that the field has been translated periodically from $[-R, R]^2$ to $[0, 2R]^2$). The region enclosed by the dotted line is what is retained as a sample of $h(x)$.

Loeve expansion with respect to the standard Fourier basis $\{e_k\}_{k\in\mathbb{K}}$ on $[-\frac{1}{2}, \frac{1}{2}]^2$:

$$u(2Ry) = \sum_{k\in\mathbb{K}} \sqrt{c(k)}\zeta(k)e_k(y), \tag{4.57}$$

where

$$c(k) = \frac{1}{(2R)^2\alpha} e^{-\frac{\pi|k|^2}{(2R)^2\alpha}},$$

and

$$\zeta(k) = \zeta_r(k) + i\zeta_i(k),$$

where $\zeta_r(k)$ and $\zeta_i(k)$ are standard Gaussian random numbers independent except for the reality constraint that $\zeta(-k) = \overline{\zeta(k)}$, for all $i \in \mathbb{K}$. By taking a finite truncation of the series over $\mathbb{K}_M = \{(k_1, k_2) \in \mathbb{K} \,|\, -M \le k_i < M, i = 1, 2\}$, then we can evaluate a realisation of u over $D_R(h)$ on a uniform $2M \times 2M$ grid using an inverse

FFT:

$$\left(u\left(\frac{Rk}{M}\right)\right)_{k\in\mathbb{K}_M} \approx FFT^{-1}\left[\left(\sqrt{c(k)}\zeta(k)\right)_{k\in\mathbb{K}_M}\right]. \tag{4.58}$$

Similarly, the derivatives $\partial_{x_j}u(x)$, $j = 1, 2$ can be computed by

$$\left(\partial_{x_j}u\left(\frac{Rk}{M}\right)\right)_{k\in\mathbb{K}_M} \approx \frac{1}{2R}FFT^{-1}\left[\left(2\pi\, i\, k_j\sqrt{c(k)}\zeta(k)\right)_{k\in\mathbb{K}_M}\right]. \tag{4.59}$$

Equation (4.59) thus provides us with a scheme to generate an approximate realisation of the derivatives of the Gaussian random field u. Implementation on a computer is then straightforward using the *gsl* implementation of the Ziggurat algorithm [Galassi and Gough, 2006; Marsaglia and Tsang, 2000] to sample $\zeta(k)$ and the FFTW library [Frigo and Johnson, 2005] to compute the inverse Fourier transform. Once a sample is generated, by discarding all but the $M \times M$ sample points which lie in B_R, we obtain an approximation to $h(x)$ over B_R. Figure 4.5 plots a realisation of $u(x)$ for $\alpha = 1$ and $R = 10$ using the method described above.

Given a realisation of the derivatives of $h(\cdot)$, we obtain the corrector χ_R by solving the following rescaled cell equation on $\mathbb{T}^2$:

$$\nabla_y\cdot\left(\sqrt{|g_R|\,(y,h)}\,{g_R}^{-1}(y,h)\left(\nabla_y\chi_R(y,h)+e\right)\right) = 0, \qquad y\in\mathbb{T}^2, \tag{4.60}$$

where $g_R(y,h) = I + \nabla h(Ry)\otimes\nabla h(Ry)$. After rescaling by R the effective diffusion coefficient $D_R(h)$ can be written as

$$D_R(h) = \frac{1}{Z_R(h)}\int_{\mathbb{T}^d}\left(I+\nabla\chi_R(y,h)\right)^* g_R^{-1}(y,h)\left(I+\nabla\chi_R(y,h)\right)\sqrt{|g_R|}\,dy, \tag{4.61}$$

where

$$Z_R(h) = \int_{\mathbb{T}^d}\sqrt{|g_R|\,(y)}\,dy.$$

For a fixed realisation h, the corrector χ_R and D_R are then approximated numerically using the finite element scheme described in Section 3.6. Computing the periodized effective diffusion coefficient we obtain a distribution of values of D_R, for which theory suggests that converges to a Dirac distribution around the actual value of D as $R\to\infty$.

Using the method described above we generate realisations of $D_R(h)$ for $\alpha = 1$ and $R\in[1,15]$ with $N = 10^3$ realisations for each R. For each realisation $h(x)$ we use the finite element scheme described in Section 3.6 to approximate $D_R(h)$. We use a starting mesh-size of 2^{-6}, stopping when the relative error of $D_R(h)$ between successive refinements is 10^{-2}. In Figure 4.6 we plot the ergodic average of samples of $D_R(h)$ for different values of R. As $N\to\infty$, the ergodic averages converges to the average value of $D_R(h)$. We note that for larger values of R the ergodic average converges very quickly. Indeed, for $R\geq 10$, the ergodic average converges to the mean after only 50 iterations. As noted in the previous example however, this comes at the cost of requiring smaller mesh-sizes to maintain a constant error for the finite element approximation as R increases. In Figure 4.7, for each R, we plot the average value of the components of $D_R(h)$. We see that there is good agreement between the mean value of $D_R(h)$ and D for large values of R and moreover that the mean value $D_R(h)$ becomes isotropic. In Figure 4.8 we plot the standard deviation of the distribution of $D_R(h)$ for $R \in [1, 15]$ and observe the variance decreasing as R increases and appears to converge to 0.

The results plotted in Figures 4.7 and 4.8 suggest that the conclusions of Theorems 4.3.3, 4.6.2 and Proposition 4.5.1 appear to hold true for the case of a Gaussian random field despite the fact that this example does not fall within the framework presented in this chapter, due to the lack of uniform bounds on the field and its derivatives.

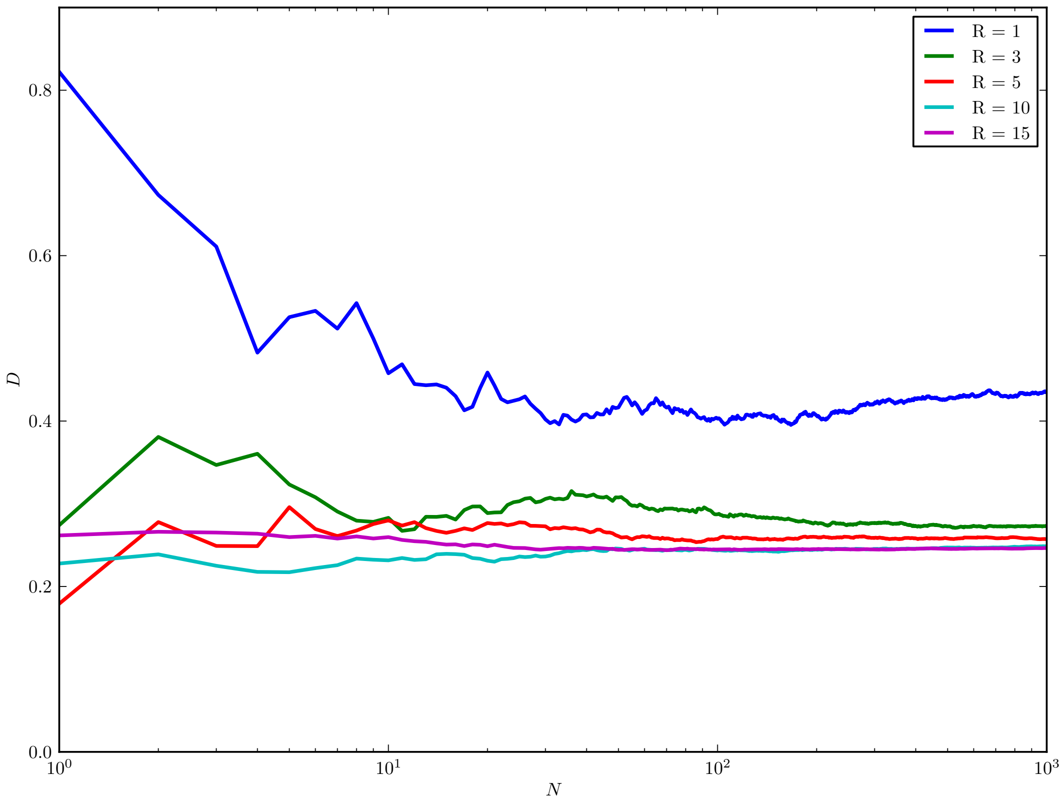


Figure 4.6: Ergodic averages of D_R for differing values of R as $N \to \infty$, for the Gaussian random field. In each case, as $N \to \infty$ converges to the average $\mathbb{E}\left[e_1 \cdot D_R(h)e_1\right]$.

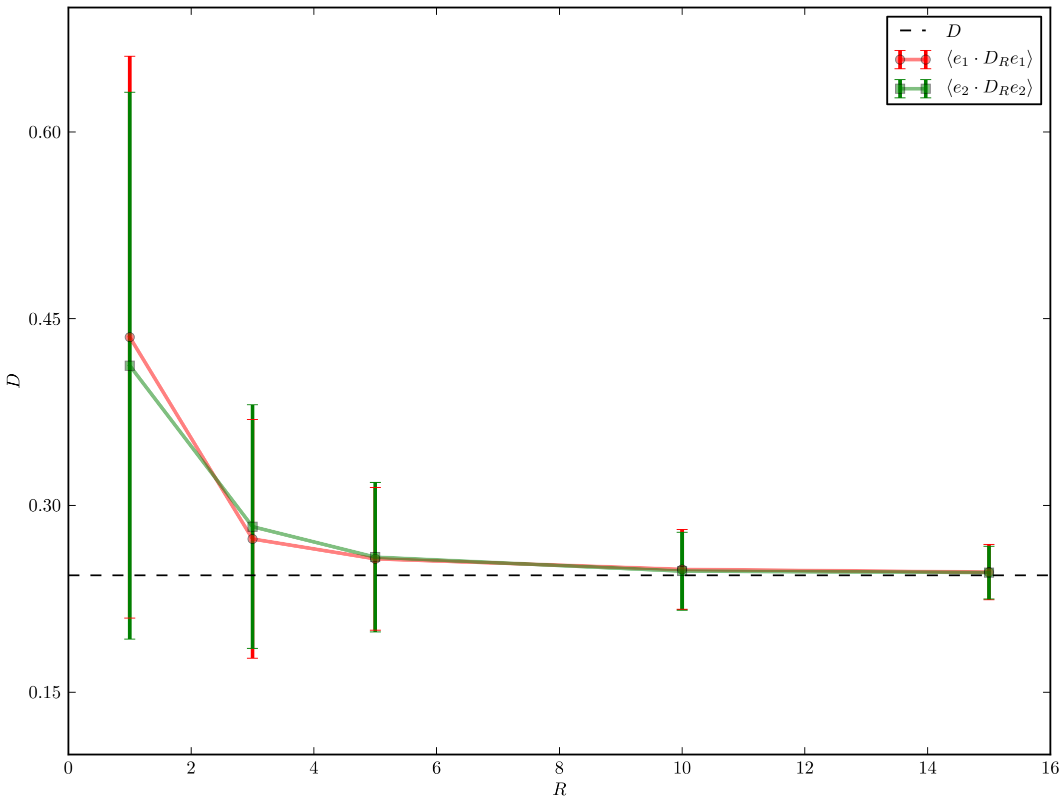
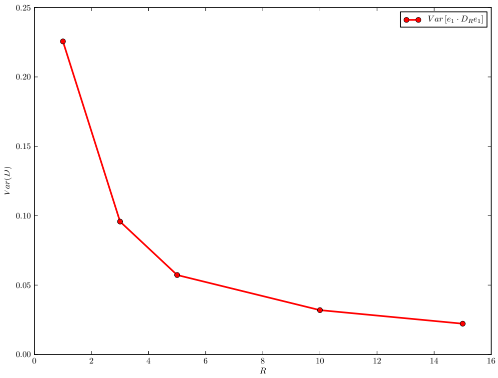
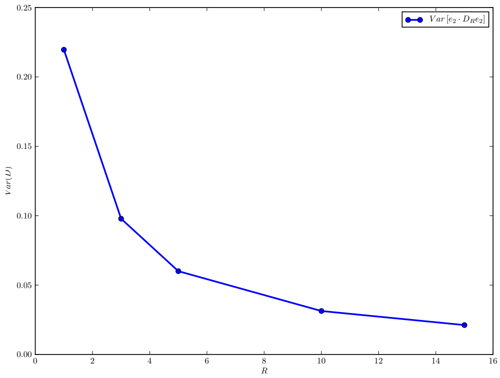


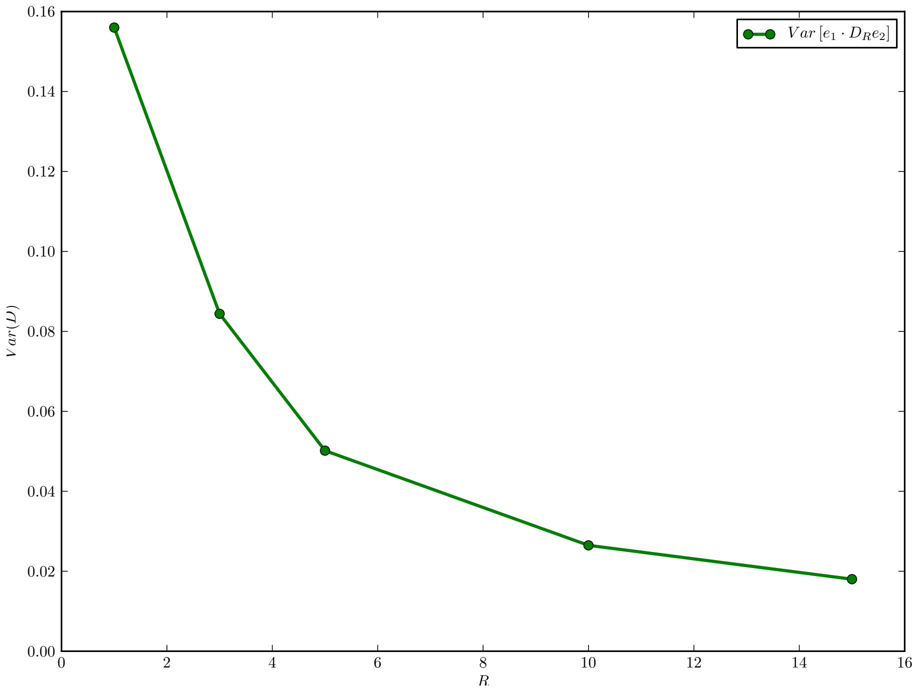
Figure 4.7: A plot of the averages of the realisations of $D_R(h)$ for increasing values of R, for the Gaussian random field surface. For each value of R, 10^3 realisations were generated. The dashed line indicates the area scaling approximation for D given by $\frac{1}{Z}\mathbf{I}$. We see that although this model does not satisfy the conditions of the homogenization theorem, homogenization does appear to occur, and we have good agreement with the theory.



(a) Plots of the standard deviation of $e_1 \cdot D_R(h)e_1$.



(b) Similar plot with $e_2 \cdot D_R(h)e_2$.



(c) Similar plot with $e_1 \cdot D_R(h)e_2$.

Figure 4.8: Plots of the standard deviation of the components of $D_R(h)$ for varying R for the Gaussian random field surface.

4.8 CONCLUSIONS AND FURTHER WORK

In this chapter we have studied the problem of diffusion on a quasi-planar surface defined by a random field which is stationary and ergodic with respect to spatial translations. We have shown that the problem of computing the effective behaviour can be expressed as a stochastic homogenization problem, and under suitable conditions on the random field, we can have applied standard results to show that the lateral diffusion process is well-approximated by a Brownian motion on the plane, with constant effective diffusion coefficient D independent of the particular surface realisation. Although D does not have a closed form for $d \geq 2$, using methods analogous to those in Chapter 3 we have been able to study various properties of D. In particular, we have derived variational bounds on the effective diffusion coefficient, showing that it is depleted with respect to the microscopic diffusion coefficient. When $d = 2$, we have been able to derive the area scaling approximation $D = \frac{1}{Z}$ for isotropic D, and provide a natural sufficient condition on the random field for D to be isotropic. We have also described a practical numerical scheme to approximate the effective diffusion coefficient using a periodic approximation, and used this method to consider two very simple examples.

There are several extensions to the present work. Clearly, as in the periodic case, it would be interesting to study the more general problem where the surface possesses a slowly varying component, and the rapid fluctuations occur normally to this slow surface. The problem of finding the effective behaviour would result in a locally-stationary homogenisation problem as was considered in [Rhodes, 2009].

Another direction of interest would be to relax Assumption $\mathbf{D}$, namely the requirement that realisations of the field and its derivatives must be uniformly bounded. Relaxing this assumption would permit one to obtain analytical results for Gaussian random fields. Removing Assumption $\mathbf{D}$ introduces several technical issues which we are unsure how to resolve. The crux of the problem lies in the fact that the drift of the SDE (4.10) is no longer bounded, and the diffusion coefficient no longer remains uniformly elliptic. This prevents us from applying Nash-Aronson estimates to obtain Gaussian bounds on the derivative of the process and thus proving the existence of a sufficiently smooth core for the generator of the environment process, $\mathcal{L}$. Without having such a core, it is impossible to prove that $\mathcal{L}$ can be written in the form (4.15), and moreover that the relation (4.16) holds for the Dirichlet form. Moreover, we are no longer justified in applying Itô's formula in the proof of Theorem 4.3.3. Nonetheless, numerical results suggest that a homogenization limit for Gaussian random fields exists, and thus we conjecture that it is possible to obtain a homogenization result for such surfaces.

Chapter 5

DIFFUSION ON TIME DEPENDENT SURFACES

In this chapter we study the scaling limits of particles diffusing laterally along the time-dependent, rapidly fluctuating surface described by the model in Section 2.3.3. As derived in Section 2.3.3, the evolution of this system is determined by the following system of Itô SDEs

$$\begin{aligned} dX^\epsilon(t) &= \frac{1}{\epsilon^\alpha} F\left(X^\epsilon(t)/\epsilon^\alpha, \eta^\epsilon(t)\right) dt + \sqrt{2\Sigma\left(X^\epsilon(t)/\epsilon^\alpha, \eta^\epsilon(t)\right)} dB(t), \\ d\eta^\epsilon(t) &= -\frac{1}{\epsilon^\beta} \Gamma \eta^\epsilon(t) dt + \sqrt{\frac{2\Gamma\Pi}{\epsilon^\beta}} dW(t), \end{aligned} \tag{S3}$$

where F and Σ are given by (2.29) and (2.30) respectively, and where the parameters α and β determine the relative scales of spatial and temporal fluctuations respectively. We identified four natural limits for this problem, Case I - IV given by:

Case I	$\alpha = 1$ and $\beta = -\infty$
Case II	$\alpha = 0$ and $\beta = 1$
Case III	$\alpha = 1$ and $\beta = 1$
Case IV	$\alpha = 1$ and $\beta = 2$

The analysis of the Case I regime, in which the surface is quenched, was performed in Section 3.8. In this chapter we will study the remaining three cases and in each case identify the asymptotic behaviour of the diffusion process, as well as study how the effective diffusion coefficient depends on the parameters of the system.

In Section 5.1 we consider the scaling limit given by Case II described in Section 2.3.3, namely where $(\alpha, \beta) = (0, 1)$. In this regime the small scale fluctuations are provided entirely by the OU process $\eta^\epsilon(t)$ for the surface modes. The limiting behaviour will be determined by the properties of the stationary distribution of the OU process $\eta^\epsilon(t)$ and deriving the effective diffusion process can be considered a straightforward averaging problem [Pavliotis and Stuart, 2008]. This regime was widely studied in the literature for the particular case of diffusion on a Helfrich elastic membrane undergoing thermal fluctuations, in particular in [Gov, 2006; Naji and Brown, 2007] and [Reister-Gottfried et al., 2007; Reister and Seifert, 2007; Reister-Gottfried et al., 2010]. To our knowledge, all previous derivations of the limiting equations have been formal. In Theorem 5.1.1 we use formal perturbation expansions to derive the limiting behaviour of the backward Kolmogorov equation corresponding to (S3), postponing a rigorous proof of the result to Appendix A.1. In Section 5.1.2 we return to the specific case where the surface is a fluctuating Helfrich membrane and obtain the corresponding limiting equations. For this particular case we then perform numerical experiments to explore the dependence of the effective diffusion coefficient on the parameters of the system.

In Section 5.2 we consider the Case III regime, in which the surface possesses both spatial and temporal fluctuations occurring at comparable scales. Although this is a natural scaling to consider for this problem, to our knowledge it has not been considered previously. We use perturbation expansions to formally derive the limiting equation for this system, and defer the rigorous proof of the homogenization result to Appendix A.2. This regime is unique in a macroscopic drift term arises in the homogenization equation. Intuition would suggest that this effective drift term is zero, however we have not been able to provide a proof of this, although we have identified a natural symmetry condition on the surface fluctuations which is sufficient to guarantee a zero macroscopic drift. In some sense, this scaling regime is intermediate between those of Case I and Case III. Indeed, in Theorem 5.2.2 we show that the effective diffusion coefficient will be given by $\overline{D}$ as defined in (3.38), so that the effective diffusion coefficient is given by the average of the (time-independent) effective diffusion equation for each realisation of the surface averaged over the stationary realisations of the surface field. Thus, the analysis of $\overline{D}$ in Section 3.8 applies here also.

Finally, in Section 5.3 we consider the Case IV scaling regime, corresponding to $(\alpha, \beta) = (1, 2)$. Obtaining the limiting dynamics in this regime is problematic due to the lack of an explicit expression for the invariant measure of the fast process, as well as the lack of uniform ellipticity of the infinitesimal generator of the fast process. Nevertheless, using the Meyn and Tweedie type argument of [Mattingly et al., 2002] and [Mattingly and Stuart, 2002] we are able to prove that the fast process is geometrically ergodic with a unique, smooth invariant density. Using this result, we provide a formal derivation of the homogenized equations and provide a rigorous proof using probabilistic methods in Appendix A.3. As the cell equation for this problem is a $K + d$ dimensional problem, a finite-element based approach is limited in this case, and so we perform numerical experiments for a simple model of lateral diffusion on a two-dimensional surface possessing a single fluctuating Fourier mode, that is, $d = 2$ and $K = 1$.

5.1 CASE II: DIFFUSION ON SURFACES POSSESSING PURELY TEMPORAL FLUCTUATIONS

We now study the case where the fast-scale fluctuations are entirely temporal, corresponding to $(\alpha, \beta) = (0, 1)$ in equation (S3). In Section 5.1.1 we use formal expansions to identify the drift and diffusion coefficients of the annealed limit process, which are given by the ergodic averages of the drift and diffusion coefficients of the multiscale problem. The subsequent sections will then focus on the Helfrich elastic model where we derive exact and asymptotic expressions for the effective diffusion coefficient providing a rigorous justification of the "preaveraging" approximations derived in [Reister and Seifert, 2007], [Gustafsson and Halle, 1997], [Naji and Brown, 2007], and others.

5.1.1 AVERAGING RESULT

In this regime, (S3) can be written as

$$\begin{aligned} dX^\epsilon(t) &= F(X^\epsilon(t), \eta^\epsilon(t))\, dt + \sqrt{2\Sigma\left(X^\epsilon(t), \eta^\epsilon\left(t\right)\right)} dB(t), \\ d\eta^\epsilon(t) &= \ -\frac{1}{\epsilon}\Gamma\eta + \sqrt{\frac{2}{\epsilon}\Gamma\Pi} dW(t), \end{aligned} \tag{5.1}$$

where F and Σ are given by (2.29) and (2.30) respectively, where Γ and Π are positive definite symmetric matrices which commute, and where $B(\cdot)$ is a standard d-dimensional Brownian motion. The process $W(\cdot)$ is a standard K-dimensional Brownian motion. The fast process $\eta^\epsilon(t)$ possesses infinitesimal generator $\frac{1}{\epsilon}\mathcal{L}_0$, where $\mathcal{L}_0$ is given by

$$\mathcal{L}_0 f(\eta) = -\Gamma\eta\nabla_\eta f(\eta) + \Gamma\Pi : \nabla_\eta\nabla_\eta f(\eta), \quad f \in C_c^2(\mathbb{R}^d).$$

The fast process $\eta^\epsilon(t)$ is geometrically ergodic with invariant distribution $\mathcal{N}(0, \Pi)$. In particular,

$$\mathcal{N}[\mathcal{L}_0] = \{\mathbf{1}\} \quad \text{and} \quad \mathcal{N}[\mathcal{L}_0^*] = \{\rho_\eta\}, \tag{5.2}$$

where

$$\rho_\eta(\eta) = \frac{1}{\sqrt{(2\pi)^d\ |\Pi|}} \exp\left(-\frac{1}{2}\eta\cdot\Pi^{-1}\eta\right). \tag{5.3}$$

The corresponding backward Kolmogorov equation for this coupled system is given by

$$\begin{aligned} \frac{\partial v^\epsilon}{\partial t}(x, \eta, t) &= \mathcal{L}^\epsilon v^\epsilon(x, \eta, t), \qquad (x, \eta, t) \in \mathbb{R}^d \times \mathbb{R}^K \times (0, T] \\ v^\epsilon(x, \eta, 0) &= v(x), \qquad x \in \mathbb{R}^d. \end{aligned} \tag{5.4}$$

where

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon}\mathcal{L}_0 + \mathcal{L}_1, \tag{5.5}$$

for

$$\mathcal{L}_1 f(x,\eta) = \frac{1}{\sqrt{|g|\,(x,\eta)}}\nabla\cdot\left(\sqrt{|g|\,(x,\eta)}g^{-1}(x)\nabla f(x,\eta)\right).$$

We now state the averaging result for this regime. We will use formal perturbation expansions to justify the result and defer the rigorous proof to Appendix A.1.

Theorem 5.1.1. *Let $T > 0$, and suppose that $\eta^\epsilon(0)$ is ρ_η-distributed. Then the process X^ϵ converges weakly in $C([0,T]\,;\mathbb{R}^d)$ to a Wiener process $X^0(t)$ which is the unique solution of the following Itô SDE in the weak sense*

$$dX^0(t) = \overline{F}(X^0(t))dt + \sqrt{2\overline{\Sigma}(X^0(t))}dB(t), \tag{5.6}$$

where

$$\overline{F}(x) = \int_{\mathbb{R}^K} F(x,\eta)\,\rho_\eta(d\eta) \tag{5.7}$$

and

$$\overline{\Sigma}(x) = \int_{\mathbb{R}^K} \Sigma(x,\eta)\rho_\eta(d\eta) \tag{5.8}$$

Moreover, assume that the backward equation (5.4) has initial data v independent of ϵ such that $v \in C_b^2(\mathbb{R}^d)$, then the solution v^ϵ of (5.4) converges pointwise to the solution v_0 of the following PDE,

$$\frac{\partial v_0}{\partial t}(x,t) = \overline{F}(x)\cdot\nabla v_0(x,t) + \overline{\Sigma}(x) : \nabla\nabla v_0(x,t).$$

uniformly with respect to t over $[0,T]$.

Formal derivation of Theorem 5.1.1. Analogous to the previous case, we look for solutions v of the form

$$v^\epsilon(x,t) = v_0(x,\eta,t) + \epsilon v_1(x,\eta,t) + \ldots,$$

for some smooth functions $v_i : \mathbb{R}^d \times \mathbb{R}^K \times [0,T] \to \mathbb{R}^d$. Substituting this ansatz in (5.4) and comparing coefficients we obtain the following pair of equations.

$O(\frac{1}{\epsilon})$**:** $\mathcal{L}_0 v_0 = 0$,

$O(1)$**:** $\frac{\partial v_0}{\partial t} = \mathcal{L}_0 v_1 + \mathcal{L}_1 v_0$.

From the $O(\frac{1}{\epsilon})$ equation and (5.2) we have that v_0 is independent of the fast-scale fluctuations. The second equation then becomes

$$\mathcal{L}_0 v_1 = \frac{\partial v_0}{\partial t} - \mathcal{L}_1 v_0.$$

Applying the Fredholm alternative, a necessary condition for the existence of a solution v_1 is that the RHS is orthogonal to the invariant measure ρ_η, that is,

$$\frac{\partial v_0}{\partial t}(x,t) = \left[\int_{\mathbb{R}^K} F(x,\eta)\rho_\eta(d\eta)\right] \cdot \nabla v_0(x,t) + \left[\int_{\mathbb{R}^K} \Sigma(x,\eta)\rho_\eta(d\eta)\right] : \nabla\nabla v_0(x,t),$$

which is the backward Kolmogorov equation for SDE (5.6). $\square$

5.1.2 DIFFUSION ON A HELFRICH SURFACE IN THE $(\alpha, \beta) = (0,1)$ REGIME

We can apply Theorem 5.1.1 to obtain the annealed limit equations for diffusion on a rapidly-fluctuating Helfrich elastic membrane. Indeed, we will show that as $\epsilon \to 0$, the process $X(\cdot)$ converges weakly to a pure diffusion process with constant diffusion coefficient. To this end, as in Section 2.2 we set $\mathbb{K} = \{k \in \mathbb{Z}^2 \setminus \{(0,0)\} \mid |k| \leq c\}$, and set the coefficients of $\eta^\epsilon(t)$ to be

$$\Gamma = \text{diag}\left(\Gamma_k\right)_{k\in\mathbb{K}} \text{ and } \Pi = \text{diag}\left(\Pi_k\right)_{k\in\mathbb{K}},$$

where Γ_k and Π_k are given by (2.15) and (2.16) respectively. The spatial functions $\{e_k\}_{k\in\mathbb{K}}$ are given by the standard $L^2(\mathbb{T}^2)$ Fourier basis: $e_k(x) = e^{2\pi i x}$. The invariant distribution of $\eta_k^\epsilon(t)$ is then given by $\mu_k = \mathcal{N}(0, \Pi_k)$.

The form of the limiting equation is strongly dependent on symmetry properties of the stationary random field.

Lemma 5.1.2. *Let $h(x)$ be a stationary realisation of the random field, that is,*

$$h(x) = \sum_{k\in\mathbb{K}} \eta_k e_k(x),$$

where $(\eta_k)_{k\in\mathbb{K}} \sim \mu_k$. Then for each $x \in \mathbb{T}^2$, the vectors

$$\left(h_{x_1}(x), h_{x_2}(x), h_{x_1x_2}(x), h_{x_1x_1}(x)\right),$$

and

$$\left(h_{x_1}(x), h_{x_2}(x), h_{x_1x_2}(x), h_{x_2x_2}(x)\right),$$

are both jointly Gaussian with mean zero, and the components of each vector are independent.

Proof. Since a finite linear combination of centered Gaussian random variables is again a centered Gaussian random variable, it is clear that both vectors are centered Gaussian random vectors. Moreover, the components of each vector are pairwise uncorrelated. To see this for $h_{x_1}(x)$ and $h_{x_2}(x)$:

$$\begin{aligned}\mathbb{E}\left[h_{x_1}(x)h_{x_2}(x)\right] &= \mathbb{E}\left[\left(\sum_{k\in\mathbb{K}}(2\pi i k_1)\eta_k e_k(x)\right)\left(\sum_{j\in\mathbb{K}}(2\pi i j_2)\eta_j e_j(x)\right)^*\right] \\ &= (2\pi)^2\sum_{k\in\mathbb{K}} k_1 k_2 \Pi_k.\end{aligned}$$

Due to the symmetry of $\mathbb{K}$ around 0, it follows that the term on the RHS is 0, so that $h_{x_1}(x)$ and $h_{x_2}(x)$ are uncorrelated. Similar arguments follow for the other pairs of components. $\square$

We now state the limit theorem for diffusion on a rapidly fluctuating Helfrich elastic membrane. This result has been stated in various previous works, in particular, in [Naji and Brown, 2007] and [Reister and Seifert, 2007].

Theorem 5.1.3. *Let $T > 0$, the process $X(\cdot)$ converges weakly in $C([0,T];\mathbb{R}^2)$, to the purely diffusive process given by*

$$dY(t) = \sqrt{2D}dB(t),$$

where $B(\cdot)$ is a standard 2D Brownian motion and $D > 0$ is given by

$$D = \frac{1}{2}\left(1 + \int_{\mathbb{R}^K}\left[\frac{1}{|g|(x,\eta)}\right]\rho_\eta(d\eta)\right). \tag{5.9}$$

Furthermore the resulting diffusion coefficient D is independent of x.

Proof. By Theorem 5.1.1, the process $X(\cdot)$ converges weakly to a process with drift coefficient $\overline{F}(x)$ and diffusion coefficient $\overline{\Sigma}(x)$ given by (5.7) and (5.8) respectively. Consider first the drift coefficient

$$\overline{F}(x) = \int_{\mathbb{R}^K}\left[\frac{(1+h_{x_1}^2)h_{x_2x_2} - 2h_{x_1}h_{x_2}h_{x_1x_2} + (1+h_{x_2}^2)h_{x_1x_1}}{(1+h_{x_1}^2+h_{x_2}^2)^2}\begin{pmatrix} h_{x_1} \\ h_{x_2}\end{pmatrix}\right]\rho_\eta(d\eta),$$

Applying the previous lemma, every term in the above sum is an odd function of a centered, Gaussian random vector. Thus each term evaluates to 0.

Similarly the effective diffusion coefficient

$$\overline{\Sigma} = \int_{\mathbb{R}^K}\left[\frac{1}{1+h_{x_1}^2+h_{x_2}^2}\begin{pmatrix} 1+h_{x_2}^2 & -h_{x_1}h_{x_2} \\ -h_{x_1}h_{x_2} & 1+h_{x_1}^2\end{pmatrix}\right]\rho_\eta(d\eta)$$

By the symmetry of h_{x_1} and h_{x_2} the off-diagonal terms also evaluate to zero, moreover, the diagonal terms are equal. Thus

$$\begin{aligned}\int_{\mathbb{R}^K}\left[\frac{1+h_{x_2}^2}{1+h_{x_1}^2+h_{x_2}^2}\right]\rho_\eta(d\eta) &= \frac{1}{2}\int_{\mathbb{R}^K}\left[\frac{1+h_{x_2}^2}{1+h_{x_1}^2+h_{x_2}^2}\right]\rho_\eta(d\eta)\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^K}\left[\frac{1+h_{x_1}^2}{1+h_{x_1}^2+h_{x_2}^2}\right]\rho_\eta(d\eta)\\ &= \frac{1}{2}\left(1+\int_{\mathbb{R}^K}\left[\frac{1}{1+h_{x_1}^2+h_{x_2}^2}\right]\rho_\eta(d\eta)\right),\end{aligned}$$

as required. Finally,

$$\begin{aligned}\nabla_x D &= \int_{\mathbb{R}^K}\nabla_x\left[\frac{1}{|g|(x,\eta)}\right]\rho_\eta(d\eta)\\ &= -2\int_{\mathbb{R}^K}\left[\frac{\nabla_x\nabla_x h(x,\eta)\nabla_x h(x,\eta)}{|g|(x,\eta)^2}\right]\rho_\eta(d\eta)=0,\end{aligned}$$

by the symmetry arguments of the previous lemma, so that D is independent of x. $\square$

Besides κ^* and σ^*, the effective diffusion coefficient also depends on the ultraviolet cut-off c (or equivalently K). One can observe that

$$\lim_{K\to\infty}\int_{\mathbb{R}^K}\left[\frac{1}{|g|(x,\eta)}\right]\rho_\eta(d\eta)=0,$$

so that for any fixed κ^*, σ^*, the effective diffusion D will approach $\frac{1}{2}$ as K approaches ∞. In fact, for fixed K, σ^* and κ^*, the effective diffusion coefficient D satisfies

$$\frac{1}{2}<D<1,$$

and recalling that the molecular diffusion coefficient D_0 was rescaled to 1, this implies that the diffusion is depleted in the limit of $\epsilon\to 0$. Although we have an explicit formula for D, we note that the ensemble average must still be computed numerically. In [Reister and Seifert, 2007], the authors derive an explicit formula for D as a function of κ^*, σ^* and K, however as noted in [Naji and Brown, 2007], they evaluate $\int_{\mathbb{R}^K}\left[\frac{1}{\int|g|(x)\,dx}\right]\rho_\eta(d\eta)$ which is significantly different from D, moreover, they do not control the errors which enter their approximation from replacing a summation with an integral. In the weak-disorder regime (i.e. when $\int|\nabla_x h(x)|^2\rho_\eta(d\eta)\ll 1$), which corresponds to the large κ^* or large σ^* regime, it is possible to derive estimates for D by applying Taylor's theorem and using the fact that $\nabla_x h(x)$ is Gaussian to get as a first order approximation,

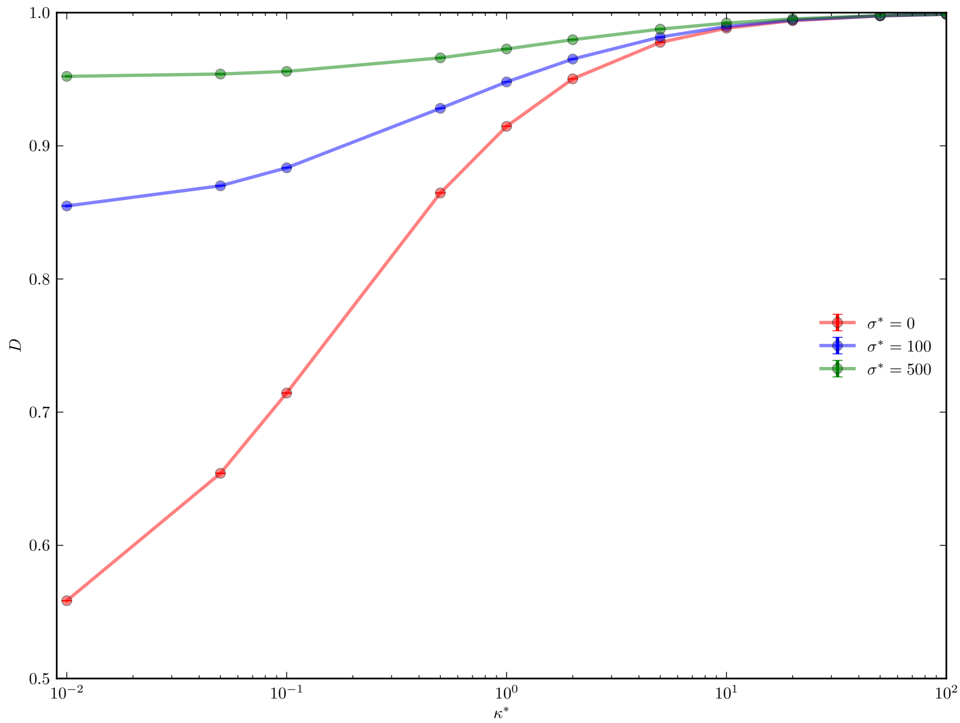
$$D=1-\frac{1}{2}\alpha^2+O(\alpha^3),$$

where α is

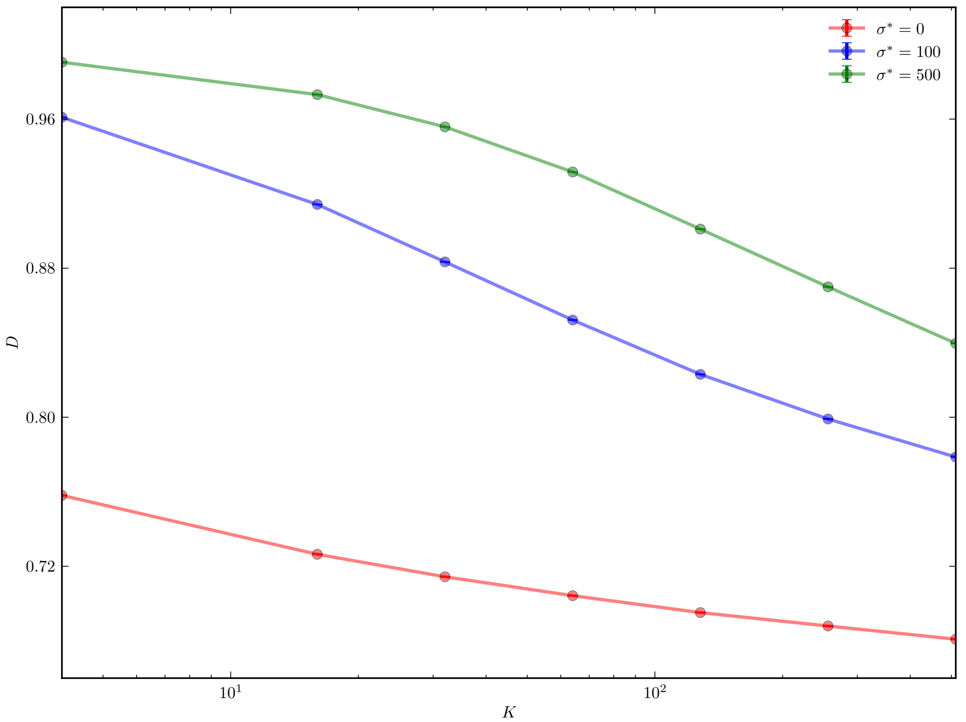
$$\alpha^2 = \sum_{k \in \mathbb{K}} \frac{1}{\kappa^* \left|2\pi k\right|^2 + \sigma^*}.$$

5.1.3 NUMERICAL EXAMPLES

We can study how the bending modulus κ^*, surface tension σ^* and the ultraviolet cut-off c (or equivalently K) affect the effective diffusion coefficient numerically. The ensemble average in (5.8) is computed using a straightforward Monte-Carlo method, taking the sample average of $\frac{1}{|g|(x,\eta)}$ for η sampled from its corresponding stationary measure. As intuition suggests, for small values of κ^* or σ^*, the larger thermal fluctuations of the surface cause a greater reduction in the speed of diffusion of a particle diffusing laterally on the surface. Indeed one can see that $(1-D) \approx \frac{1}{\kappa^*}$ for fixed σ^* and $(1-D) \approx \frac{1}{\sigma^*}$ for fixed κ^*. In Figure 5.1 we plot D for varying values of κ^*, K and for $\sigma^* = 0, 100, 500$. The convergence of D to $\frac{1}{2}$ becomes immediately apparent. As expected, D decays logarithmically with c, converging to $\frac{1}{2}$ as $c \to \infty$.



(a) The effective diffusion coefficient for a diffusion on a Helfrich surface computed for various values of bending rigidity κ and surface tension σ, for $K = 32$.



(b) The effective diffusion coefficient for a diffusion on a Helfrich surface for varying K.

Figure 5.1: The effective diffusion coefficient for a diffusion on a fluctuating Helfrich elastic membrane in the $(0, 1)$ scaling.

5.2 CASE III: DIFFUSION ON SURFACES WITH COMPARABLE SPATIAL AND TEMPORAL FLUCTUATIONS

We now consider the $(\alpha, \beta) = (1, 1)$ scaling which describes lateral diffusion on a rough surface which is also fluctuating rapidly, but the temporal surface fluctuations occur slower than the characteristic scale of the spatial fluctuations. This scaling limit was considered for SDEs with periodic spatial and temporal fluctuations in [Garnier, 1997]. A unique characteristic of this scaling regime is that it gives rise to a macroscopic drift term in the limit as $\epsilon \to 0$ which is determined by the rate of change of the corrector with respect to the temporal fluctuations.

A similar effective drift term arises in the model considered here. Although intuition suggests that this effective drift will be zero, it is not clear that this is the case, and indeed we cannot prove this in general. However, we identify a natural symmetry condition for the surface fluctuations for which we can prove the effective drift term is zero. In Section 5.2.1 we identify the coefficients of the limiting diffusion equation. In Section 5.2.2 we study the properties of the effective diffusion and provide sufficient conditions for the isotropy of the effective diffusion coefficient and for the effective drift to be zero. In Section 5.2.3 we study the limiting properties of the Helfrich model in this regime.

5.2.1 HOMOGENIZATION RESULT

Introducing the fast process $Y^\epsilon(t) = \frac{X^\epsilon(t)}{\epsilon} \mod \mathbb{T}^d$, equations (S3) can be written as the following fast-slow system

$$\begin{aligned}
dX^\epsilon(t) &= \frac{1}{\epsilon} F\left(Y^\epsilon(t), \eta^\epsilon(t)\right) dt + \sqrt{2\Sigma\left(Y^\epsilon(t), \eta^\epsilon(t)\right)} dB(t), \\
dY^\epsilon(t) &= \frac{1}{\epsilon^2} F\left(Y^\epsilon(t), \eta^\epsilon(t)\right) dt + \sqrt{\frac{2}{\epsilon^2}\Sigma\left(Y^\epsilon(t), \eta^\epsilon(t)\right)} dB(t), \\
d\eta^\epsilon(t) &= -\frac{1}{\epsilon}\Gamma\eta^\epsilon(t)dt + \sqrt{\frac{2\Gamma\Pi}{\epsilon}} dW(t),
\end{aligned} \tag{5.10}$$

where F and Σ are given by (2.29) and (2.30) respectively and where we impose periodic boundary conditions on $Y^\epsilon(\cdot)$. The processes $B(\cdot)$ and $W(\cdot)$ are standard d and K-dimensional Brownian motions, respectively. The infinitesimal generator of the fast process $Y^\epsilon(t)$ is given by $\frac{1}{\epsilon^2}\mathcal{L}_0$, where $\mathcal{L}_0$ is given by

$$\mathcal{L}_0 f(y) = \frac{1}{\sqrt{|g|(y,\eta)}} \nabla_y \cdot \left(\sqrt{|g|(y,\eta)} g^{-1}(y,\eta) \nabla_y f(y)\right), \quad f \in C_c^2(\mathbb{T}^d). \tag{5.11}$$

We note that although the spatial and temporal fluctuations appear commensurate in the system of SDEs, the spatial fluctuations relax to equilibrium at a timescale faster than the temporal fluctuations. The limiting equation can thus be considered the result of a reiterated homogenisation/averaging problem of the form described in Section 2.11.3 of [Bensoussan et al., 1978]. The limiting equation is thus obtained by homogenising over $Y^\epsilon(t)$ for a frozen value of $\eta^\epsilon(t)$ and then averaging over the invariant measure $\rho_\eta(\cdot)$ of $\eta^\epsilon(t)$.

For η fixed $\mathcal{L}_0$ satisfies

$$\mathcal{N}[\mathcal{L}_0] = \{\mathbf{1}\} \qquad \text{and} \qquad \mathcal{N}[\mathcal{L}_0^*] = \{\rho_y(y,\eta)\},$$

where $\mathcal{L}_0^*$ is the formal adjoint of $\mathcal{L}_0$, and where

$$\rho_y(y,\eta) = \frac{\sqrt{|g(y,\eta)|}}{Z(\eta)}$$

for $Z(\eta) = \int_{\mathbb{T}^2} \sqrt{|g(y,\eta)|}\, dy$.

For $\eta \in \mathbb{R}^K$ fixed, we choose the corrector $\chi(y,\eta)$ to be the solution of the following cell equation

$$\mathcal{L}_0\chi(y,\eta) = -F(y,\eta), \qquad (y,\eta) \in \mathbb{T}^d \times \mathbb{R}^K. \tag{5.12}$$

Lemma 5.2.1. *There exists a unique solution $\chi \in C^{2\times 2}(\mathbb{T}^d \times \mathbb{R}^K; \mathbb{R}^d)$ such that*

$$\int_{\mathbb{T}^d} \chi(y,\eta)\rho_y(dy,\eta) = 0, \qquad \eta \in \mathbb{R}^K, \tag{5.13}$$

and which solves (5.12).

Proof. Since, for each $\eta \in \mathbb{R}^K$, the centering condition

$$\int_{\mathbb{T}^d} F(y,\eta)\rho(y,\eta)\, dy = 0,$$

we can use the Fredholm alternative as in Lemma 3.2.1 to prove the existence of a unique solution $\chi(y,\eta)$. The regularity of the corrector in y and η follows from a straightforward bootstrap argument. □

The backward Kolmogorov equation corresponding to the coupled system (5.10) is given by

$$\begin{aligned} \frac{\partial v^\epsilon}{\partial t}(x,y,\eta,t) &= \mathcal{L}^\epsilon v^\epsilon(x,y,\eta,t), \qquad (x,y,\eta,t) \in \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{R}^K \times (0,T] \\ v^\epsilon(x,0) &= v(x), \end{aligned} \tag{5.14}$$

where

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon^2}\mathcal{L}_0 + \frac{1}{\epsilon}\mathcal{L}_\eta + \frac{1}{\epsilon}\mathcal{L}_1 + \mathcal{L}_2, \tag{5.15}$$

for

$$\mathcal{L}_1 f(x,y,\eta) = F(y,\eta)\cdot\nabla_x f(x,y,\eta) + 2\Sigma(y,\eta) : \nabla_x\nabla_y f(x,y,\eta),$$

and

$$\mathcal{L}_2 f(x,y,\eta) = \Sigma(y,\eta) : \nabla_x\nabla_x f(x,y,\eta),$$

and $\mathcal{L}_\eta$ is the infinitesimal generator of the OU process and is given by

$$\mathcal{L}_\eta f(\eta) = -\Gamma \cdot \nabla_\eta f(\eta) + \Gamma\Pi : \nabla_\eta \nabla_\eta f(\eta). \tag{5.16}$$

We note that the $2\Sigma(y,\eta) : \nabla_x \nabla_y f$ term arises due to the correlation between the noise in the slow processes.

We assume that the initial condition v is independent of the fast process. We can now state the homogenisation result for this scaling and provide a formal justification using perturbation expansions. A rigorous proof based on probabilistic methods will be given in Appendix A.2. As in the previous chapters we use the convention that $\left(\nabla_y \chi\right)_{ij} = \frac{\partial \chi^{e_i}}{\partial y_j}$.

Theorem 5.2.2. *Let $0 < \epsilon \ll 1$ and $T = \mathcal{O}(1)$, and suppose $\eta^\epsilon(0)$ is ρ_η-distributed, where ρ_η is given by (5.3). Then as $\epsilon \to 0$, the process $X^\epsilon(\cdot)$ converges weakly in $C([0,T];\mathbb{R}^d)$ to X^0 which is a weak solution of the following Itô SDE*

$$dX^0(t) = L dt + \sqrt{2D} dB(t), \tag{5.17}$$

where the effective diffusion coefficient D is given by

$$D = \int_{\mathbb{R}^K} \int_{\mathbb{T}^d} \left(I + \nabla_y \chi\right) g^{-1}(y,\eta) \left(I + \nabla_y \chi\right)^\top \rho_y(y,\eta)\, \rho_\eta(\eta)\, dy d\eta, \tag{5.18}$$

and the effective drift term L is given by

$$L = \int_{\mathbb{R}^K} \int_{\mathbb{T}^d} \mathcal{L}_\eta \chi \, \rho_y(y,\eta)\, \rho_\eta(\eta)\, dy d\eta. \tag{5.19}$$

Moreover, if the backward Kolmogorov equation (5.14) has initial data v (independent of ϵ) such that $v \in C_b^2(\mathbb{R}^d)$, then the solution v^ϵ of (5.14) converges pointwise to the solution v_0 of

$$\frac{\partial v_0}{\partial t}(x,t) = L \cdot \nabla_x v_0(x,t) + D : \nabla_x \nabla_x v_0(x,t), \tag{5.20}$$

uniformly with respect to t over $[0,T]$.

Formal derivation of Theorem 5.2.2. We make the ansatz that

$$v^\epsilon = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots,$$

for some smooth functions $v_i : \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{R}^K \times [0,T] \to \mathbb{R}$. Substituting v^ϵ in (5.14) and equating identical powers of ϵ we obtain the following equations:

$O(\frac{1}{\epsilon^2}) : \; \mathcal{L}_0 v_0 = 0,$

$O(\frac{1}{\epsilon}) : \; \mathcal{L}_0 v_1 = -\mathcal{L}_\eta v_0 - \mathcal{L}_1 v_0,$

$O(1) : \; \mathcal{L}_0 v_2 = -\left(\frac{\partial v_0}{\partial t} - \mathcal{L}_\eta v_1 - \mathcal{L}_1 v_1 - \mathcal{L}_2 v_0\right).$

The first equation implies that $v_0 \in \mathcal{N}[\mathcal{L}_0]$ so that v_0 is a constant in y. The second equation thus becomes

$$\mathcal{L}_0 v_1 = \left(\mathcal{L}_\eta v_0 + F(y,\eta)\cdot\nabla_x v_0\right).$$

By the Fredholm alternative applied to $\mathcal{L}_0$ a necessary and sufficient condition for the existence of a solution v_1 is that the RHS is centered with respect to $\sqrt{|g|}$, for each fixed x and η that is,

$$\int_{\mathbb{T}^d}\left(F(y,\eta)\cdot\nabla_x v_0 + \mathcal{L}_\eta v_0\right)\sqrt{|g|\,(y,\eta)}\,dy = 0.$$

The first term in the above integral is clearly 0. Since $\mathcal{L}_\eta v_0$ is independent of y centering condition becomes

$$Z(\eta)\mathcal{L}_\eta v_0 = 0.$$

Since $Z > 1$, it must follow that $v_0 \in \mathcal{N}[\mathcal{L}_\eta]$. By the ergodicity of the Ornstein Uhlenbeck process $\eta(t)$ over $\mathbb{R}^K$ it follows that v_0 is also independent of η so that v_0 is a function of x only. The second equation thus becomes

$$\mathcal{L}_0 v_1 = F(y,\eta)\cdot\nabla_x v_0.$$

Let $\chi(\cdot,\eta)$ be the unique solution of the cell equation (5.12) which is guaranteed by Lemma 5.2.1, then choosing $v_1 = \chi\cdot\nabla_x v_0$ it is clear that v_1 solves the $O(\frac{1}{\epsilon})$ equation.

We now consider the $O(1)$ equation. By the Fredholm alternative, a necessary condition for the existence of a unique solution v_2 is that the RHS is centered with respect to the invariant measure of $\mathcal{L}_0$. That is,

$$\frac{\partial v_0}{\partial t} = \int_{\mathbb{T}^2}\left(\mathcal{L}_\eta v_1 + \mathcal{L}_1 v_1 + \mathcal{L}_2 v_0\right)\rho_y(y,\eta)\,dy,$$

which, substituting the definitions of the $\mathcal{L}_i$'s and v_j's, can be written as follows

$$\begin{aligned}\frac{\partial v_0}{\partial t} &= \int_{\mathbb{T}^d} F(y,\eta)\otimes\chi(y,\eta)\,\rho_y(y,\eta)\,dy : \nabla_x\nabla_x v_0 \\ &+ \int_{\mathbb{T}^d} g^{-1}(y,\eta)\nabla_y\chi^\top(y,\eta)\rho_y(y,\eta) + \nabla_y\chi(y,\eta)\,g^{-1}(y,\eta)\rho_y(y,\eta)\,dy : \nabla_x\nabla_x v_0 \\ &+ \int_{\mathbb{T}^d} g^{-1}(y,\eta)\rho_y(y,\eta)\,dy : \nabla_x\nabla_x v_0 \\ &+ \int_{\mathbb{T}^d}\mathcal{L}_\eta\chi(y,\eta)\rho_y(y,\eta)\,dy\nabla_x v_0.\end{aligned} \tag{5.21}$$

First we note that

$$\int_{\mathbb{T}^d} F(y,\eta) \otimes \chi(y,\eta)\sqrt{|g|}\,dy = -\int_{\mathbb{T}^d} \mathcal{L}_0\chi(y,\eta) \otimes \chi\sqrt{|g|\,(y,\eta)}\,dy$$
$$= \int_{\mathbb{T}^d} \nabla_y\chi(y,\eta) g^{-1}(y,\eta)\nabla_y\chi^\top(y,\eta)\sqrt{|g|\,(y,\eta)}\,dy,$$

so that we can write (5.21) as

$$\frac{\partial v_0}{\partial t} = \int_{\mathbb{T}^d} \left(I + \nabla_y\chi(y,\eta)\right) g^{-1}(y,\eta) \left(I + \nabla_y\chi(y,\eta)\right)^\top \rho_y(y,\eta)\,dy : \nabla_x\nabla_x v_0$$
$$+ \int_{\mathbb{T}^d} \mathcal{L}_\eta\chi(y,\eta)\rho_y(y,\eta)\,dy\nabla_x v_0.$$

Averaging with respect to the invariant measure ρ_η of $\mathcal{L}_1$ we derive the effective diffusion equation

$$\frac{\partial v_0}{\partial t} = \int\int_{\mathbb{T}^d} \left(I + \nabla_y\chi(y,\eta)\right) g^{-1}(y,\eta) \left(I + \nabla_y\chi(y,\eta)\right)^\top \rho_y(y,\eta)\,\rho_\eta(\eta)\,dy\,d\eta : \nabla_x\nabla_x v_0$$
$$+ \int\int_{\mathbb{T}^d} \mathcal{L}_\eta\chi\rho_y(y,\eta)\rho_\eta(\eta)\,dy\,d\eta \cdot \nabla_x v_0,$$

or more compactly

$$\frac{\partial v_0(x,t)}{\partial t} = D : \nabla_x\nabla_x v_0(x,t) + L \cdot \nabla_x v_0(x,t),$$

where D and L are given by (5.18) and (5.19) respectively. The above formal expansions suggest that the process $X^\epsilon(\cdot)$ converges weakly to a process $X^0(\cdot)$ given by the solution of Itô SDE (5.17). We shall prove this rigorously in Appendix A.3. □

This homogenised diffusion equation agrees with the result derived in Section 6.1 of [Garnier, 1997], in particular Example 6.1 in which the author proves a homogenisation result for a diffusion in a potential V which is a function of a single 1-dimensional OU process.

5.2.2 PROPERTIES OF THE EFFECTIVE DIFFUSION PROCESS

Comparing the effective behaviour of the homogenized diffusion processes in Case I and Case III, we see that the introduction of the fast temporal fluctuations gives rise to a time-averaging of the effective diffusion coefficient, so that the effective diffusion in the $(\alpha,\beta) = (1,1)$ case is equal to $\overline{D}$, the averaged effective diffusion coefficient for diffusion on a surface with quenched fluctuations, as described in Section 3.8. Thus all the properties proved for $\overline{D}$ hold equally for the effective diffusion coefficient D. The following proposition summarizes the most important properties.

Proposition 5.2.3. *Let D be the effective diffusion given by (5.18), then*

(i) D is a symmetric, positive definite matrix.

(ii) In particular, for a unit vector $e \in \mathbb{R}^d$

$$0 < e \cdot \overline{D_*}\, e \le e \cdot De \le e \cdot \overline{D^*} e \le 1, \tag{5.22}$$

where $\overline{D^*} = \mathbb{E}\left[D^*(h)\right]$ and $\overline{D_*} = \mathbb{E}\left[D_*(h)\right]$, for D^* and D_* given by (3.23) and (3.22) respectively, and where $\mathbb{E}\left[\cdot\right]$ denotes expectation with respect to the invariant measure of $\eta^\epsilon(t)$.

(iii) For $d = 2$, if the condition of Proposition 3.8.1 holds, then D is isotropic.

(iv) If, additionally $\mathbb{E}\left[|\nabla h(x)|^2\right] = \delta \ll 1$, then

$$D = \overline{D_{as}} + O(\delta^4),$$

where $D_{as} = \mathbb{E}\left[\frac{1}{Z(h)}\right]$.

□

We turn our attention to the effective drift term L given by (5.19). Unlike D, the effective drift depends on $\chi(y, \eta)$ which is only unique up to a constant depending on η. However, for any function $c(\eta)$ we have that

$$\begin{aligned}\int \left[\int_{\mathbb{T}^d} \left(\mathcal{L}_\eta c(\eta)\right) \rho_y(y,\eta)\rho_\eta(\eta)\, dy\right] d\eta &= \int \mathcal{L}_\eta c(\eta) \left(\int_{\mathbb{T}^d} \rho_y(y,\eta)\, dy\right) \rho_\eta(\eta)\, d\eta \\ &= \int \mathcal{L}_\eta c(\eta)\rho_\eta(\eta) d\eta = 0,\end{aligned}$$

since $\int_{\mathbb{T}^d} \rho_y(y,\eta)\, dy = 1$ for all η. It follows that the effective drift L is uniquely defined independent of any additive terms independent of y.

The fact that an effective drift would arise in this scaling limit is surprising and intuitively we would expect it to be 0, however we have not been able to prove this in general. Numerical simulations suggest that L is always zero. To this end, we make the following conjecture.

Conjecture 5.2.1. *The effective drift coefficient L is zero.*

□

While we are not able to prove this conjecture in general we have been able to show that it holds for a large class of surfaces, which satisfy a natural symmetry condition.

The assumption we make is the following. Suppose there exists a linear orthogonal map $\mathcal{C} : \mathbb{R}^K \to \mathbb{R}^K$ which commutes with Π and Γ (in particular ρ_η is invariant with respect to $\mathcal{C}$) such that

$$h(x, \mathcal{C}^\top \eta) = h(x^\perp, \eta), \tag{5.23}$$

where $x_i^\perp = 1 - x_i$ for $i \in \{1, \dots d\}$, or equivalently that

$$\mathcal{C} e(x) = e(x^\perp),$$

where $e(x) = \{e_k(x)\}_{k \in \mathbb{K}}$.

Condition (5.23) arises naturally in the case where e_k are the Fourier basis for the Laplacian on $[0,1]^2$. The surface perturbation h can then be rewritten as

$$h(x, \eta) = \sum_{k \in \mathbb{K}_{even}} \eta_k^e e_k^e(x) + \sum_{k \in \mathbb{K}_{odd}} \eta_k^o e_k^o(x),$$

where e_k^e and e_k^o are respectively even and odd functions on $[0,1]^2$ for all $k \in \mathbb{K}$. If $\mathcal{C}$ is the diagonal matrix defined by

$$\mathcal{C}^\top \eta = \mathcal{C}^\top (\eta^e, \eta^o) = (\eta^e, -\eta^o),$$

for $\eta^e = (\eta_k^e)_{k \in \mathbb{K}_{even}}$ and $\eta^o = (\eta_k^o)_{k \in \mathbb{K}_{odd}}$, we see that condition (5.23) is trivially satisfied. More generally, Condition (5.23) will typically hold for surfaces possessing some form of symmetry in the fluctuations. We can show the following result

Proposition 5.2.4. *Suppose (5.23) holds, then the effective drift coefficient L is zero.*

Proof. We first note that (5.23) implies that

$$g^{-1}(x, \mathcal{C}^\top \eta) = g^{-1}(x^\perp, \eta),$$

and

$$|g|\,(x, \mathcal{C}^\top \eta) = |g|\,(x^\perp, \eta).$$

Consider the cell equation for the corrector $\chi^e(y, \eta)$ given by

$$\nabla \cdot \left(\sqrt{|g|\,(y, \eta)} g^{-1}(y, \eta) \left(\nabla \chi^e(y, \eta) + e \right) \right) = 0.$$

Making the substitution $\eta \to \mathcal{C}\eta$, then using the relations for g^{-1} and $|g|$ and changing variables in y we have

$$-\nabla \cdot \left(\sqrt{|g|\,(y, \eta)} g^{-1}(y, \eta) \left(-\nabla \tilde{\chi}^e(y, \eta) + e \right) \right) \Bigg|_{y = y^\perp}, \tag{5.24}$$

where $\tilde{\chi}^e(y, \eta) = \chi^e(y^\perp, \mathcal{C}\eta)$. It follows that

$$\chi^e(y^\perp, \mathcal{C}\eta) = -\chi^e(y, \eta). \tag{5.25}$$

Applying (5.24) and using the fact that $\mathcal{C}$ commutes with Γ and Π, we obtain

$$\begin{aligned}
-\mathcal{L}_\eta \chi^e(y^\perp, \eta) &= -\Gamma\eta \cdot \mathcal{C}\nabla_\eta \chi^e(y, \mathcal{C}^\top \eta) + \mathcal{C}^\top \Gamma\Pi\mathcal{C} : \nabla\nabla\chi^e(y, \mathcal{C}^\top\eta) \\
&= -\Gamma\mathcal{C}^\top \eta \cdot \nabla_\eta \chi^e(y, \mathcal{C}^\top\eta) + \Gamma\Pi : \nabla\nabla\chi^e(y, \mathcal{C}^\top\eta) \\
&= \mathcal{L}_\eta \chi^e(y, \mathcal{C}^\top \eta).
\end{aligned}$$

Using the invariance of ρ_y with respect to $\mathcal{C}$ the effective drift term V will then be given by

$$\begin{aligned}
L &= \int\int_{\mathbb{T}^d} \mathcal{L}_\eta \chi^e(y^\perp, \eta)\rho_y(y^\perp, \eta)\rho_\eta(\eta)\, dyd\eta \\
&= -\int\int_{\mathbb{T}^d} \mathcal{L}_\eta \chi^e(y, \mathcal{C}^\top\eta)\rho_y(y, \mathcal{C}^\top\eta)\rho_\eta(\eta)\, dyd\eta \\
&= -L,
\end{aligned}$$

proving the result. □

5.2.3 DIFFUSIONS ON HELFRICH SURFACES IN THE $(\alpha, \beta) = (1, 1)$ REGIME

Using the results of the previous section, we can study the limiting behaviour of diffusion on a two-dimensional fluctuating Helfrich surface in the $(\alpha, \beta) = (1, 1)$ scaling. Following the discussion subsequent to Conjecture 5.2.1 we see that the conditions of Proposition 5.2.4 hold, and so that the effective drift is $\mathbf{0}$. As the effective diffusion D is equal to the averaged diffusion coefficient $\overline{D}$ of Section 3.8.1, the dependence of D on the parameters κ^*, σ^* and K hold equivalently.

5.3 CASE IV: DIFFUSION ON SURFACES WITH TEMPORAL FLUCTUATIONS FASTER THAN SPATIAL FLUCTUATIONS

In this Section we consider the $(\alpha, \beta) = (1, 2)$ scaling. In this scaling the surface possesses rapid spatial and temporal fluctuations but the temporal fluctuations much faster than the spatial fluctuations. Writing $Y^\epsilon(t) := \frac{X^\epsilon(t)}{\epsilon} \mod \mathbb{T}^d$ the fast-slow system for this regime is given by:

$$\begin{aligned}
dX^\epsilon(t) &= \frac{1}{\epsilon} F\left(Y^\epsilon(t), \eta^\epsilon(t)\right)\, dt + \sqrt{2\Sigma\left(Y^\epsilon(t), \eta^\epsilon(t)\right)} dB(t), \\
dY^\epsilon(t) &= \frac{1}{\epsilon^2} F\left(Y^\epsilon(t), \eta^\epsilon(t)\right)\, dt + \sqrt{\frac{2}{\epsilon^2}\Sigma\left(Y^\epsilon(t), \eta^\epsilon(t)\right)} dB(t), \\
d\eta^\epsilon(t) &= -\frac{1}{\epsilon^2}\Gamma\eta^\epsilon(t)\, dt + \sqrt{\frac{2\Gamma\Pi}{\epsilon^2}} dW(t),
\end{aligned} \tag{5.26}$$

where we impose periodic boundary conditions on $Y^\epsilon(\cdot)$, and where $B(\cdot)$ is a standard d-dimensional Brownian motion, and $W(\cdot)$ is a standard K-dimensional Brownian motion. The infinitesimal generator of the underlying fast process is given by

$\frac{1}{\epsilon^2}\mathcal{G}$ where

$$\mathcal{G}f(y,\eta) = \left(\mathcal{L}_0 + \mathcal{L}_\eta\right) f(y,\eta), \qquad f \in C_c^2(\mathbb{T}^d \times \mathbb{R}^K),$$

where $\mathcal{L}_0$ and $\mathcal{L}_\eta$ are given by (5.11) and (5.16) respectively. We note that although the temporal fluctuations occur at a faster timescale to the spatial fluctuations, the spatial and temporal fluctuations relax to equilibrium at a comparable timescale.

Unlike in the previous cases, it is not immediately clear that the fast process is geometrically ergodic. Moreover, due to the unbounded support of the surface fluctuations, the infinitesimal generator is no longer uniformly elliptic. Thus we cannot apply standard elliptic theory to obtain a Fredholm alternative for this operator. In Proposition 5.3.1 we prove that the fast process possesses a unique invariant measure and moreover that the convergence to equilibrium is exponentially fast. The proof, given in Appendix (A.3) is a straightforward application of the results in the papers [Mattingly et al., 2002; Mattingly and Stuart, 2002] which are based on the results of [Meyn and Tweedie, 1993]. In Proposition 5.3.1 we show that there exists a unique, smooth solution of the cell equation for this scaling limit, provided the cell equation holds.

5.3.1 HOMOGENIZATION RESULT

We first identify the fast process $(Y^\epsilon(t), \eta^\epsilon(t))$ as a rescaling of a $\mathbb{T}^d \times \mathbb{R}^K$-valued process independent of ϵ. Indeed, define $(Y(t), \eta(t))$ as follows

$$\begin{aligned} dY(t) &= F(Y(t),\eta(t))dt + \sqrt{2\Sigma\left(Y(t),\eta(t)\right)}\, d\hat{B}(t), \\ d\eta(t) &= -\Gamma\eta(t) + \sqrt{2\Gamma\Pi}\, d\hat{W}(t), \end{aligned} \tag{5.27}$$

where $\hat{B}(t)$ is a standard $\mathbb{R}^d$-valued Brownian motion, $\hat{W}(t)$ is a standard $\mathbb{R}^K$-valued Brownian motion and where we impose periodic boundary conditions on $Y(t)$ with period 1. The joint process $(Y(t), \eta(t))$ has infinitesimal generator $\mathcal{G}$. It is straightforward to show that the following equality holds (in law),

$$(Y^\epsilon(t), \eta^\epsilon(t)) = \left(Y(t/\epsilon^2), \eta(t/\epsilon^2)\right).$$

Proposition 5.3.1. *The process $(Y(t), \eta(t))$ possesses a unique invariant measure ρ with smooth, positive density with respect to the Lebesgue measure on $\mathbb{T}^d \times \mathbb{R}^K$ which is the unique, normalised solution of*

$$\mathcal{G}^*\rho = 0. \tag{5.28}$$

Let $P(t)$ be the Markov semigroup induced by $(Y(t), \eta(t))$. Then there exists a constant $\mu \in (0,1)$ such that for all functions $f : \mathbb{T}^d \times \mathbb{R}^K \to \mathbb{R}$, such that

$$|f|(y,\eta) \leq CV(\eta), \qquad (y,\eta) \in \mathbb{T}^d \times \mathbb{R}^K, \tag{5.29}$$

where

$$V(\eta) := (1 + |\eta|^2) \tag{5.30}$$

the following estimate holds,

$$\left|\mathbb{E}^{(y_0,\eta_0)} f(Y(t), \eta(t)) - \int f(y,\eta)\rho(dy,d\eta)\right| \le C' V(\eta_0) e^{-\mu t}, \tag{5.31}$$

where $\mathbb{E}^{(y_0,\eta_0)}$ *denotes expectation conditioned on* $(Y(0), \eta(0)) = (y_0, \eta_0) \in \mathbb{T}^d \times \mathbb{R}^K$. *In particular, this implies that*

$$\left\| P(t)f - \int f(y,\eta)\rho(dy,d\eta) \right\|_{L^2(\rho)} \le C'' e^{-\mu t}, \tag{5.32}$$

for some positive constants C', C''.

The cell equation for this scaling limit takes the following form

$$\mathcal{G}\chi(y,\eta) = -F(y,\eta), \qquad (y,\eta) \in \mathbb{T}^d \times \mathbb{R}^K. \tag{5.33}$$

The existence of a smooth solution χ to equation (5.33) is guaranteed by the following result.

Proposition 5.3.2. *Suppose the following centering assumption holds*

$$\int_{\mathbb{T}^d \times \mathbb{R}^K} F(y,\eta)\,\rho(dy,d\eta) = 0. \tag{5.34}$$

Then there exists a unique, smooth solution $\chi \in D(\mathcal{G})$ *such that*

$$\int_{\mathbb{T}^d \times \mathbb{R}^K} \chi(y,\eta)\,\rho(dy,d\eta) = 0$$

which solves (5.33). The solution χ *satisfies*

$$|\chi(y,\eta)| \le C(1 + |\eta|^2), \qquad (y,\eta) \in \mathbb{T}^d \times \mathbb{R}^K$$

where $C > 0$ *is a constant independent of* (y,η). *Moreover,*

$$\begin{aligned}\int \nabla_y \chi^\top g^{-1} \nabla_y \chi\,\rho(dy,d\eta) + \int \nabla_\eta \chi(y,\eta)^\top\,\Gamma\Pi\,\nabla_\eta\chi(y,\eta)\,\rho(dy,d\eta)& \\ = -2\int \chi(y,\eta) \otimes \mathcal{G}\chi(y,\eta)\,\rho(dy,d\eta) < \infty.&\end{aligned} \tag{5.35}$$

Proof. This is a direct application of Proposition A.3.1 from Appendix A.3 applied to $b(y,\eta) = F(y,\eta)\cdot e$, for a general unit vector $e \in \mathbb{R}^d$. □

As before, the backward Kolmogorov equation corresponding to (5.26) is given by

$$\frac{\partial v^\epsilon}{\partial t}(x,y,\eta,t) = \mathcal{L}^\epsilon v^\epsilon(x,y,\eta,t), \qquad (x,y,\eta,t) \in \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{R}^K \times (0,T]$$
$$v^\epsilon(x,0) = v(x), \tag{5.36}$$

where

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon^2}\mathcal{G} + \frac{1}{\epsilon}\mathcal{L}_1 + \mathcal{L}_2 \tag{5.37}$$

for

$$\begin{aligned}\mathcal{L}_1 f(x,y,\eta) &= \frac{1}{\sqrt{|g|\,(y,\eta)}}\nabla\cdot\left(\sqrt{|g|\,(y,\eta)}g^{-1}(y,\eta)\right)\cdot\nabla_x f(x,y,\eta)\\ &\quad + 2g^{-1}(y,\eta) : \nabla_x\nabla_y f(x,y,\eta),\\ \text{and } \mathcal{L}_2 f(x,y,\eta) &= g^{-1}(y,\eta) : \nabla_x\nabla_x f(x,y,\eta).\end{aligned}$$

As in the previous cases, we note that the mixed derivative term $2g^{-1} : \nabla_x\nabla_y f$ arises due to the correlation between noise in the fast and slow processes.

We assume that the initial condition ϕ is independent of the fast processes. Having Propositions 5.3.1 and 5.3.2 we can state the homogenization result for this regime. We provide a formal derivation based on multiscale expansions here. A rigorous proof of the invariance prinicple using Itô's formula and a martingale central limit theorem can be found in Section A.3.

Theorem 5.3.3. *Suppose Assumption (5.34) holds and $\eta^\epsilon(0)$ is ρ_η-distributed, where ρ_η is given by (5.3). Let $0 < \epsilon \ll 1$ and $T = \mathcal{O}(1)$. Then as $\epsilon \to 0$, the process $X^\epsilon(\cdot)$ converges weakly in $C([0,T];\mathbb{R}^d)$ to the process $X^0(\cdot)$ which is the weak solution of the following Itô SDE*

$$dX^0(t) = \sqrt{2D}dB(t), \tag{5.38}$$

where the effective diffusion coefficient D is given by

$$D = \int \left(I + \nabla_y\chi\right) g^{-1}\left(I + \nabla_y\chi\right)^\top \rho(dy,d\eta) + \int \nabla_\eta\chi\,\Gamma\Pi\nabla_\eta\chi^\top\,\rho(dy,d\eta). \tag{5.39}$$

Moreover, if the backward equation (5.36) has initial data v (independent of ϵ) such that $v \in C_b^2(\mathbb{R}^d)$, then the solution v^ϵ of (5.36) converges pointwise to the solution v_0 of

$$\frac{\partial v_0}{\partial t}(x,t) = D : \nabla_x\nabla_x v_0(x,t), \tag{5.40}$$

uniformly with respect to t over $[0,T]$.

Formal derivation of Theorem 5.3.3. We look for solutions v of the form

$$v^\epsilon = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \dots$$

of (5.36) for some smooth functions $v_i : \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{R}^K \times [0,T] \to \mathbb{R}$. Substituting this ansatz in (5.36) and equating equal powers of ϵ we obtain the following three equations

$$O(\tfrac{1}{\epsilon^2}) : \mathcal{G} v_0 = 0,$$

$$O(\tfrac{1}{\epsilon}) : \mathcal{G} v_1 = -\mathcal{L}_1 v_0,$$

$$O(1) : \mathcal{G} v_2 = -\left(\tfrac{\partial v_0}{\partial t} - \mathcal{L}_1 v_1 - \mathcal{L}_2 v_0\right).$$

As the fast process is ergodic, the nullspace of $\mathcal{G}$ is spanned by constant functions. Thus, the $O\left(\frac{1}{\epsilon^2}\right)$ implies that v_0 is independent of y and η. The second equation thus becomes

$$\mathcal{G} v_1 = -F(y,\eta) \cdot \nabla_x v_0.$$

Since we are assuming Assumption (5.34), there exists a unique solution of the cell equation (5.33), by Proposition 5.3.2. By choosing $v_1 = \chi \cdot \nabla_x v_0$ we see that the second equation is satisfied.

By Proposition A.3.1, a sufficient condition for the final equation to have a solution is that the RHS is orthogonal to the measure $\rho(dy, d\eta)$, (assuming that the RHS grows at most polynomially). That is,

$$\begin{aligned}\frac{\partial v_0}{\partial t}(y,\eta) = \int F(y,\eta) \cdot \nabla_x v_1 \, \rho(dy, d\eta) + \int 2\Sigma(y,\eta) : \nabla_x \nabla_y v_1 \, \rho(dy, d\eta) \\ + \int \Sigma(y,\eta) : \nabla_x \nabla_x v_0 \, \rho(dy, d\eta),\end{aligned}$$

which we can rewrite as

$$\frac{\partial v_0}{\partial t} = D : \nabla_x \nabla_x v_0,$$

where the effective diffusion coefficient D is given by

$$D = \int \left[\frac{1}{\sqrt{|g|}} \nabla_y \cdot \left(\sqrt{|g|} g^{-1}\right) \otimes \chi + g^{-1} \nabla_y \chi^\top + \nabla_y \chi g^{-1} + g^{-1} \right] \rho(dy, d\eta)$$

Note that the first term on the RHS

$$\int \frac{1}{\sqrt{|g|}} \nabla_y \cdot \left(\sqrt{|g|} g^{-1}\right) \otimes \chi \, \rho(dy, d\eta) : \nabla_x \nabla_x v_0,$$

can be rewritten as $\mathcal{K} : \nabla_x \nabla_x v_0$, where

$$\mathcal{K} = \mathrm{Sym}\left[\int \frac{1}{\sqrt{|g|}} \nabla_y \cdot \left(\sqrt{|g|} g^{-1}\right) \otimes \chi \, \rho(dy, d\eta) \right],$$

where Sym $[\cdot]$ denotes the symmetric part of the matrix. Let $e \in \mathbb{R}^d$ be a unit vector and consider

$$\mathcal{K}^e := e \cdot \mathcal{K} e = \int \frac{1}{\sqrt{|g|}} \nabla_y \cdot \left(\sqrt{|g|} g^{-1} e\right) \chi^e \, \rho(dy, d\eta),$$

where $\chi^e = \chi \cdot e$. Noting that

$$-\mathcal{G}\chi^e = \frac{1}{\sqrt{|g|}} \nabla_y \cdot \left(\sqrt{|g|} g^{-1} e\right),$$

it follows that

$$\mathcal{K}^e = \int \chi^e \left(-\mathcal{G}\chi^e\right) \rho(dy, d\eta),$$

which, by (5.35) can be written as

$$\mathcal{K}^e = \int \nabla_y \chi^e \cdot g^{-1} \nabla_y \chi^e \, \rho(dy, d\eta) + \int \nabla_\eta \chi^e \cdot \Gamma \Pi \nabla_\eta \chi^e \, \rho(dy, d\eta),$$

so that

$$e \cdot De = \int \left(e + \nabla_y \chi^e\right) \cdot g^{-1} \left(e + \nabla_y \chi^e\right) \rho(dy, d\eta) + \int \nabla_\eta \chi^e \cdot \Gamma \Pi \nabla_\eta \chi^e \, \rho(dy, d\eta),$$

or in matrix notation

$$D = \int \left(I + \nabla_y \chi\right) g^{-1} \left(I + \nabla_y \chi\right)^\top \rho(dy, d\eta) + \int \nabla_\eta \chi \, \Gamma \Pi \nabla_\eta \chi^\top \, \rho(dy, d\eta),$$

The above formal computations suggest that the process $X^\epsilon(t)$ converges weakly to a process $X^0(\cdot)$ given by the solution of the Itô SDE given by (5.38). □

A question which has not been addressed is the validity of the centering condition. Due to the lack of an explicit invariant measure for the fast process it is not clear whether or not the centering condition holds. Numerical experiments suggest that the centering condition does hold in general, however we have not been able to prove this. We make the following conjecture:

Conjecture 5.3.1. *The centering condition (5.34) holds in general.*

□

We are, however, able to show that the centering condition will hold if the the symmetry condition (5.23) holds.

Proposition 5.3.4. *Suppose equation (5.23) holds then the centering condition (5.34) holds.*

Proof. We first note that condition (5.23) implies that

$$g^{-1}(y, \mathcal{C}\eta) = g^{-1}(y^\perp, \eta),$$

and

$$|g|\,(y, \mathcal{C}\eta) = |g|\,(y^\perp, \eta),$$

and moreover that

$$F(y, \mathcal{C}\eta) = -F(y^{\perp}, \eta). \tag{5.41}$$

Consider the equation for the invariant density ρ given by

$$\mathcal{G}^*\rho(y,\eta) = \nabla_y \cdot \left(\sqrt{|g|\,(y,\eta)}\, g^{-1}(y,\eta)\nabla_y \left(\frac{\rho(y,\eta)}{\sqrt{|g|\,(y,\eta)}} \right) \right) + \nabla_\eta \cdot \left(\rho_\eta(\eta)\Gamma\Pi\nabla_\eta \left(\frac{\rho(y,\eta)}{\rho_\eta(\eta)} \right) \right) = 0.$$

Making the substitution $\eta \to \mathcal{C}\eta$, then using the relations for g^{-1} and $|g|$, the invariance of ρ_η with respect to $\mathcal{C}$ and the fact that Γ, Π and $\mathcal{C}$ commute

$$\nabla_y \cdot \left(\sqrt{|g|\,(y^\top,\eta)}\, g^{-1}(y^\top,\eta)\nabla_y \left(\frac{\tilde{\rho}(y^\top,\eta)}{\sqrt{|g|\,(y^\top,\eta)}} \right) \right) + \nabla_\eta \cdot \left(\rho_\eta(\eta)\Gamma\Pi\nabla_\eta \left(\frac{\tilde{\rho}(y^\top,\eta)}{\rho_\eta(\eta)} \right) \right) = 0,$$

where $\tilde{\rho}(y,\eta) := \rho(y^\top, \mathcal{C}\eta)$. It follows that

$$\rho(y^\top, \mathcal{C}\eta) = \rho(y,\eta), \qquad (y,\eta) \in \mathbb{T}^d \times \mathbb{R}^K.$$

Consider the centering term. Changing variables $\eta \to \mathcal{C}^\top\eta$ in the integral we get that

$$\begin{aligned} \hat{F} &= \int_{\mathbb{T}^d}\int_{\mathbb{R}^K} F(y,\eta)\rho(y,\eta)\,dy\,d\eta \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{R}^K} F(y,\mathcal{C}^\top\eta)\rho(y,\mathcal{C}^\top\eta)\,dy\,d\eta \\ &= -\int_{\mathbb{T}^d}\int_{\mathbb{R}^K} F(y^\top,\eta)\rho(y^\top,\eta)\,dy\,d\eta. \end{aligned}$$

Changing variables $y \to y^\perp$ in the last equality we thus get that $\hat{F} = -\hat{F}$, so that the centering condition holds. □

5.3.2 NUMERICAL SIMULATIONS FOR HELFRICH SURFACES IN THE $(\alpha, \beta) = (1, 2)$ REGIME

Is it straightforward to see that the effective diffusion coefficient is both symmetric and positive definite. Due to the lack of an explicit invariant measure for the fast process, extracting further properties and bounds from the effective diffusion coefficient is not straightforward. To obtain some intuition about the effective behaviour in this scaling we do numerical simulations to compute the effective diffusion coefficient D. Rather than resort to direct numerical simulations of the coupled SDEs, we instead use a finite element scheme to solve the equations for the invariant measure and the cell equation. The finite element approximation then becomes $K+2$ dimensional problem. For the sake of tractability, we restrict our interest to when $d=2$ and when the surface consists of a single Fourier mode, i.e. $K=1$. To compute D numerically we performed the following steps:

1. We formulated a piecewise linear finite element approximation to equation (5.28) on a regular, triangulated mesh of the domain

$$\Omega_M = \{(y_1, y_2, \eta) \in [0,1] \times [0,1] \times [-M, M]\},$$

   where M is chosen so that the support of ρ outside $[-M, M]$ is small. We imposed periodic boundary conditions on the boundaries in the y_1 and y_2 directions, and purely reflective boundary conditions in the η direction. Together, boundary conditions ensure that probability is conserved.

2. The solution ρ of (5.33) is then obtained by solving the corresponding generalised eigenvalue problem for the 0 eigenvector. The resulting eigenvector was then normalised over Ω_M to give an approximation to ρ.

3. The components of the corrector χ^{e_1} and χ^{e_2} were then computed by solving the cell equation (5.33) using a piecewise linear finite element scheme on the same mesh.

4. Finally, the components of the effective diffusion coefficient were then computed by integrating (5.39) using standard quadrature over Ω_M.

As the distribution ρ has unbounded support, care must be taken in choosing the domain length M to be sufficiently large to obtain an accurate approximation of ρ. The marginal distribution of $\rho(dy, d\eta)$ over $y \in \mathbb{T}^2$ is given by $\mathcal{N}(0, \Pi)$, where $\Pi > 0$ is the (scalar) variance of the invariant measure of $\eta(t)$. It thus follows that the region $[-M, M]$ will contain probability mass $\operatorname{erf}\left(\frac{M}{\sqrt{2\Pi}}\right)$. Therefore, choosing $M > 2^{\frac{3}{2}}\Pi$ will ensure that approximately 99.5% of the density is contained in the region $[-M, M]$.

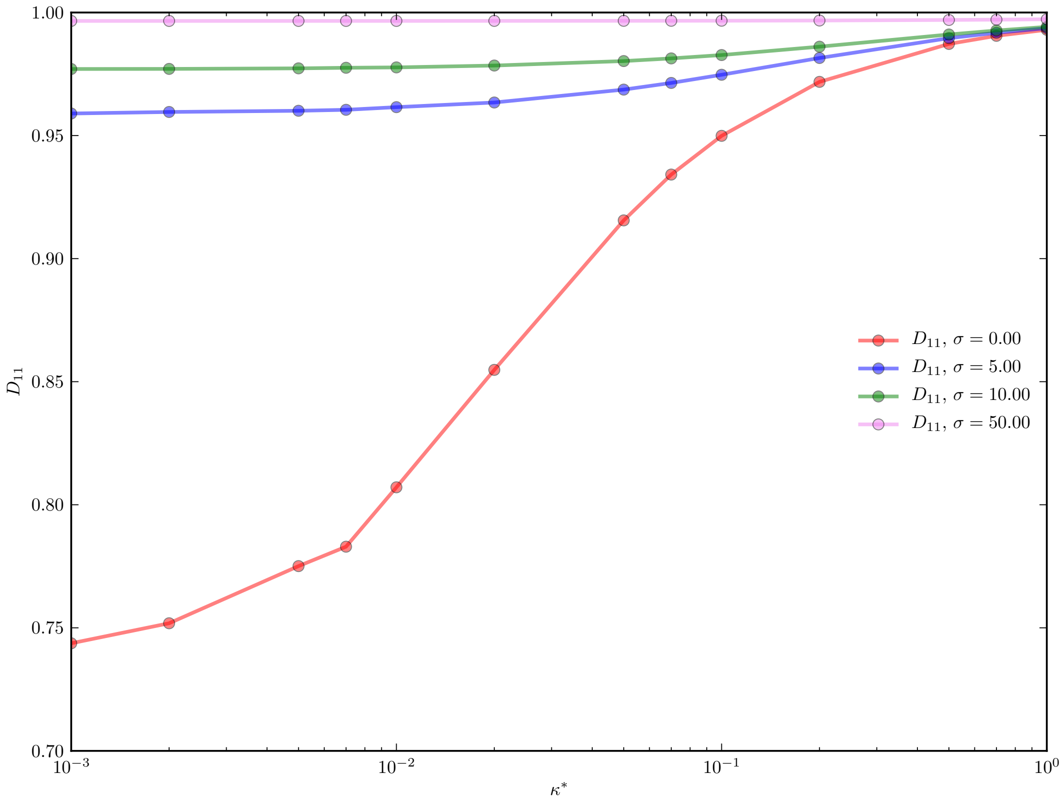
We applied the above steps to compute the effective diffusion coefficient for the surface given by $h(x, \eta(t))$ where

$$h(x, \eta) = \eta \sin(2\pi x) \sin(2\pi y), \tag{5.42}$$

and $\eta(t)$ is an OU process with SDE

$$d\eta(t) = -\eta(t)\, dt + \sqrt{\frac{2}{\kappa^* |2\pi k|^4 + \sigma^* |2\pi k|^2}}\, dW(t).$$

In Figure 5.2 we plot the first diagonal component of D for $\kappa^* \in [0.05, 5]$ and for $\sigma^* = 0, 5, 10$ and 50, respectively. We note immediately that the symmetry in $h(x, \eta)$ is sufficient to ensure that D is isotropic. Moreover, as in the previous scaling limits being considered, D is bounded above by 1, so that the macroscopic diffusion is depleted with respect to the molecular diffusion coefficient.



(a) Plots of the first component of the effective diffusion coefficient D for varying κ^* and σ^*

Figure 5.2: Plots of the components of the effective diffusion coefficient for the surface given by (5.42) in the $(\alpha, \beta) = (1, 2)$ regime.

5.4 CONCLUSIONS AND FURTHER WORK

The problem of lateral diffusion on a random surface possessing both rapid spatial and temporal fluctuations has been studied in this chapter. For four particular scaling regimes Case II to Case IV, we have used formal multiscale expansions to show that the effective behaviour of the laterally diffusing process can be well-approximated by a Brownian motion on the plane, with a constant diffusion coefficient D. Rigorous proofs of the convergence results are defered to Appendix A.

In the Case II scaling regime, corresponding to $\alpha = 0$, $\beta = 1$, we have derived the effective equation, and for the particular case of the Helfrich elastic surface, we have shown that the effective equation agrees with the “pre-averaging approximation" of [Reister and Seifert, 2007], as well as the results in [Halle and Gustafsson, 1997; Naji and Brown, 2007; Gov, 2006]. We have then studied the dependence of the effective diffusion coefficient on the model parameters κ^* and σ^* via numerical experiments.

In the Case III scaling regime, corresponding to $\alpha = 1$, $\beta = 1$, we have shown that the effective diffusion coefficient equals $\overline{D}$ as defined in Section 3.8, namely the average effective diffusion coefficient for a stationary realisation of the surface, averaged over all realisations of the surface. Thus all all the properties of Section 3.8 apply in this case also.

Finally, we have looked at the Case IV scaling regime, where $\alpha = 1$ and $\beta = 2$, regime. The analysis of this scaling limit is considerably more involved due to the lack of an explicit invariant measure for the fast process. Applying a Meyn and Tweedie technique, based on [Mattingly et al., 2002; Mattingly and Stuart, 2002] we have been able to prove existence of a unique invariant measure, and that the fast process is geometrically ergodic with respect to this measure. Using this fact, along with Hörmander's theorem, we have shown that the corresponding cell equation is well posed, so that standard formal multiscale expansions can be used to prove the homogenization theorem. Due to the high-dimension of the fast process and the unboundedness of the domain, solving the cell equation directly is expensive. Nonetheless, for a very simple example with $d = 2$ and $K = 1$, we have used a standard 3 dimensional piecewise finite element scheme to compute the effective diffusion.

There are numerous issues and potential further directions which can be explored:

1. We are not able to find a general proof to Conjecture 5.2.1, although we have proved it to hold for a large class of functions. We suspect that a proof will be forthcoming if we study the symmetries in y and η which are preserved by the solution of the cell equation.

2. Similarly, we are unsure whether or not the cell equation holds in general for the Case IV scaling regime. The two conjectures are closely related and we suspect that a proof of one would lead to a proof of the other since both problems involve studying transformations under which the operator $\mathcal{L}_\eta$ is invariant.

3. It would be interesting to consider extensions to the simple model (S3) which we considered in this chapter. One interesting direction would be to incorporate non-thermal fluctuations. The dynamics of active membrane have been widely studied (see for example, [Lacoste and Lau, 2005; Lin et al., 2006; Loubet et al., 2012; Naji et al., 2009]). At the most basic level, the dynamics of an active membrane could be described by augmenting the SPDE for a thermally-fluctuating Helfrich membrane, given in (2.10), with an additional noise term having positive characteristic correlation time τ_a. The spectrum of the active noise term would be determined according to the nature of the active force (e.g. direct or curvature type forcings, [Loubet et al., 2012]). One could study the effect on the effective diffusion coefficient induced by the correlation time and spectrum of these active fluctuations.

4. Another area which has recieved much attention (see [Reister-Gottfried et al., 2010]) is that of curvature-coupled diffusion, where the particle is not only

constrained to the membrane, but also induces a change in the membrane shape by inducing a spontaneous curvature at its current location. Following [Leitenberger et al., 2008; Naji et al., 2009] one can formulate a very simple model of this problem, by augmenting the Helfrich free energy (2.8) with an additional energy term which depends on both the membrane configuration and the position of the particle. The effective dynamics of such a system prove to be very interesting, with the fast dynamics (membrane fluctuations) depending on the slow process (particle position).

Chapter 6

OTHER SCALING LIMITS FOR THE HELFRICH ELASTIC SURFACE

In Chapter 5 we investigated the limiting behaviour of $X^\epsilon(t)$ given by (S3) for four particular scaling regimes, namely Cases I - IV corresponding to $(\alpha, \beta) = (1, -\infty)$, $(0, 1)$, $(1, 1)$ and $(1, 2)$, respectively. While these four regimes seem "representative" of the different possible scalings of the system, we did not address the issue of the limiting behaviour outside these regimes.

While it is possible to formulate (S3) for any choice of α and β, without further assumptions on the structure of the spatial functions $e(x)$ and the drift and diffusion coefficients of the OU process Γ and Π, not every choice of α and β will give rise to a limit as $\epsilon \to 0$, although, in these cases it is possible to subtract out terms of the form $O(\frac{1}{\epsilon^\gamma})$ to obtain a finite limit as $\epsilon \to 0$.

However, for the model of lateral diffusion on a thermally fluctuating Helfrich surface given in Section 2.2, it is possible to exhaustively recover all the distinguished limits of the system (with one exception). Indeed we can show that the limits obtained in the Case I - IV regimes are the *only* possible distinguished limits of the system.

To this end we consider the joint process $(X^\epsilon(t), \eta^\epsilon(t))$ given by (S3) where the coefficients Γ and Π of $\eta^\epsilon(t)$ are given by (2.15) and (2.16) respectively, and where $e_k(x) = e^{2\pi i k\cdot x}$. The objective of this chapter is to derive the limiting equations for $X^\epsilon(t)$ for every distinct scaling determined by α and β. Following the proofs of Chapter 5, we adopt a probabilistic approach to deriving these limits.

The methods used in the proof are based upon those used in Chapter 3.10 of [Bensoussan et al., 1978], where the similar problem of homogenization of an SDE with coefficients which are periodic in both space and time is studied. However, due to a number of errors made in the derivation, the authors omit a number of drift terms which arise in the homogenization limit. These omissions were then corrected in

[Garnier, 1997]. We note that, unlike these two works, we have the added complication of the unbounded Gaussian fluctuations of the OU process, although this obstacle is easily overcome given the results in light of the results proved in Appendix A.2.

We first note that by relabeling ϵ^α appropriately, we need only consider the following scaling regimes

$$(\alpha,\beta) = (0,1) \qquad \text{and} \qquad (\alpha,\beta) \in \{1\} \times [-\infty,\infty).$$

The results which we prove in the remainder of the chapter are summarised in the following result.

Theorem 6.0.1. *Denote by $D_1(\eta)$, D_2, D_3 and D_4 the effective diffusion coefficients given by (3.12), (5.9), (5.18) and (5.39) respectively, corresponding to Case I to IV. Then, the process $X^\epsilon(t)$ converges weakly in $C([0,T];\mathbb{R}^2)$ to the process $X^0(t)$ according to the following table*

(i)	$\alpha = 0$, $\beta = 1$,	$X^0(t) = \sqrt{2D_2}B(t)$
(ii)	$\alpha = 1$, $-\infty \leq \beta < 0$,	$X^0(t) = \sqrt{2D_1(\eta(0))}B(t)$
(iii)	$\alpha = 1$ *and* $\beta = 0$,	$X^0(t) = \sqrt{2D_1(\eta(t))}B(t)$
(iv)	$\alpha = 1$ *and* $0 < \beta < 2$,	$X^0(t) = \sqrt{2D_3}B(t)$
(v)	$\alpha = 1$ *and* $\beta = 2$,	$X^0(t) = \sqrt{2D_4}(t)$
(vi)	$\alpha = 1$ *and* $2 < \beta \leq 3$,	*Not determined*
(vii)	$\alpha = 1$ *and* $\beta > 3$,	$X^0(t) = \sqrt{2D_2}B(t)$

□

Remark Note that the regime $\alpha = 1$, $2 < \beta \leq 3$ has been omitted since one requires additional compatibility conditions to ensure the existence of a limiting PDE, and it is not clear whether these hold, even for the case of diffusion on a fluctuating Helfrich surface.

As can be seen, the different effective behaviour can be broadly split into three separate classes depending on the relative speed of the spatial and temporal fluctuations. For $\beta \leq 0$ the fast fluctuations are contributed entirely by the small scale spatial structure of the surface and no averaging over the fluctuating surface modes occurs. This regime can thus be considered to be a trivial extension of the scaling limit derived in Case I. If the relaxation time of the Fourier modes is comparable with the timescale of the lateral diffusion process then the effective diffusion coefficient will depend on the current state $\eta(t)$ of the surface. If the surface is fluctuating at an incomparably smaller timescale, then at the $O(1)$ timescale the surface is quenched and the effective diffusion coefficient will depend only on the initial surface configuration.

For $0 < \beta < 2$, the OU process will relax to equilibrium sufficiently fast for averaging to occur at $O(1)$ scales. At an $O(\frac{1}{\epsilon})$ timescale, the process will have homogenized over the spatial fluctuations for a 'frozen' surface configuration. At the $O(1)$

timescale additional averaging will take place due to temporal fluctuations. The effective diffusion coefficient D_3 will be the spatially homogenized diffusion coefficient $D_1(\eta)$ averaged over the invariant measure of $\eta(t)$, as was described in Case III.

For $\beta > 3$ the rapid temporal fluctuations dominate the fast process, and the diffusion process will have been averaged over the surface Fourier modes even at the characteristic timescale of the rapid spatial fluctuations. Thus over macroscopic timescales the diffusion process is well-approximated by its annealed disorder limit, as in Case II for $(\alpha, \beta) = (0, 1)$.

In the remainder of this chapter we prove Theorem 6.0.1. Cases (i) and (v) have already been considered in Chapter 5. The proofs for Cases (ii) and (iii) are trivial modifications of the proof for the Case I regime. We prove the remaining in the following sections.

6.1 PROOF FOR $\alpha = 1$ AND $0 < \beta < 1$

For fixed $\eta \in \mathbb{R}^K$, define $\chi : \mathbb{T}^2 \times \mathbb{R}^K \to \mathbb{R}^2$ to be the unique solution of

$$\mathcal{L}_y \chi(y, \eta) = -F(y, \eta), \qquad (y, \eta) \in \mathbb{T}^2 \times \mathbb{R}^K,$$

such that $\int_{\mathbb{T}^2} \chi(y, \eta)\, \rho_y(y, \eta)\, dy = 0$ for all $\eta \in \mathbb{R}^K$. By Itô's formula

$$\begin{aligned}
\epsilon^{-1} F(Y^\epsilon(t), \eta^\epsilon(t)) &= \epsilon \left(\chi(Y^\epsilon(0), \eta^\epsilon(0)) - \chi(Y^\epsilon(t), \eta^\epsilon(t)) \right) \\
&\quad + \int_0^t \sqrt{2\Sigma(Y^\epsilon(s), \eta^\epsilon(s))} \nabla_y \chi(Y^\epsilon(s), \eta^\epsilon(s))\, dB(s) \\
&\quad + \epsilon^{1-\beta} \int_0^t \mathcal{L}_\eta \chi(Y^\epsilon(s), \eta^\epsilon(s))\, ds \\
&\quad + \epsilon^{1-\beta/2} \int_0^t \sqrt{2\Gamma\Pi} \nabla_\eta \chi(Y^\epsilon(s), \eta^\epsilon(s))\, dW(s),
\end{aligned}$$

so that

$$\begin{aligned}
X^\epsilon(t) - X^\epsilon(0) &= \epsilon \left(\chi(Y^\epsilon(0), \eta^\epsilon(0)) - \chi(Y^\epsilon(t), \eta^\epsilon(t)) \right) \\
&\quad + \int_0^t \sqrt{2\Sigma(Y^\epsilon(s), \eta^\epsilon(s))} \left(I + \nabla_y \chi(Y^\epsilon(s), \eta^\epsilon(s)) \right) dB(s) \\
&\quad + \epsilon^{1-\beta} \int_0^t \mathcal{L}_\eta \chi(Y^\epsilon(s), \eta^\epsilon(s))\, ds \\
&\quad + \epsilon^{1-\beta/2} \int_0^t \sqrt{2\Gamma\Pi} \nabla_\eta \chi(Y^\epsilon(s), \eta^\epsilon(s))\, dW(s) \\
&= M^\epsilon(t) + R^\epsilon(t),
\end{aligned}$$

where $M^\epsilon(t)$ is a martingale term, and $R^\epsilon(t)$ is a remainder term which is $O(\epsilon^{1-\beta})$. Using the ergodicity of the fast process, it is straightforward to show the quadratic variation of $M^\epsilon(t)$ is given by

$$[\![M^\epsilon]\!] = \int_0^t (I + \nabla_y \chi(Y^\epsilon(s), \eta^\epsilon(s)))\Sigma(Y^\epsilon(s), \eta^\epsilon(s)) \left(I + \nabla_y \chi(Y^\epsilon(s), \eta^\epsilon(s))\right)^\top \, ds,$$

converges to

$$D = \int_{\mathbb{R}^K} \int_{\mathbb{T}^2} \left(I + \nabla_y \chi(y, \eta)\right) \Sigma(y, \eta) \left(I + \nabla_y \chi(y, \eta)\right)^\top \rho_y(y, \eta) \rho_\eta(\eta) \, dy \, d\eta,$$

as $\epsilon \to 0$.

It follows the martingale central limit theorem [Ethier and Kurtz, 2009, Theorem 7.1.4] that $M^\epsilon(t)$ converges weakly in $C([0,T]; \mathbb{R}^2)$ to a Brownian motion $\sqrt{2D}\, B(t)$. Moreover, arguing as in the proof of Theorem 5.2.2 one can show that the remainder term $R^\epsilon(t)$ converges weakly in $C([0,T]; \mathbb{R}^2)$ to 0, thus proving the result for this regime.

6.2 PROOF FOR $\alpha = 1$ AND $1 < \beta < 2$

We shall see that the limiting diffusion equation in this regime will infact be identical to that of Case III. This property is specific to this particular model. Indeed, as described in [Garnier, 1997], certain values of β in the range $1 < \beta < 2$ would give rise to large effective drifts which must be subtracted out for the scaling limit to exist.

To ensure that any effective drifts are zero we make a symmetry assumption on the random field h, namely that condition (5.23) of Section 5.2.3 holds.

The following lemma will then be used to ensure that any effective drifts which arise in the limiting equation are 0.

Lemma 6.2.1. *Assume condition (5.23) holds. Let $H : \mathbb{T}^2 \times \mathbb{R}^K \to \mathbb{R}$ such that $\int_{\mathbb{T}^2} H(y, \eta) \rho_y(y, \eta) \, dy = 0$ for all $\eta \in \mathbb{R}^K$, and suppose moreover that*

$$H(y, \mathcal{C}\eta) = -H(y^\perp, \eta). \tag{6.1}$$

Let χ be the unique solution of

$$-\mathcal{L}_y \chi(y, \eta) = H(y, \eta), \qquad (y, \eta) \in \mathbb{T}^2 \times \mathbb{R}^K,$$

such that $\int_{\mathbb{T}^2} \chi(y, \eta) \rho_y(y, \eta) \, dy = 0$ then

$$\chi(y, \mathcal{C}\eta) = -\chi(y^\perp, \mathcal{C}\eta), \qquad (y, \eta) \in \mathbb{T}^2 \times \mathbb{R}^K$$

and

$$\int_{\mathbb{R}^K} \int_{\mathbb{T}^2} \mathcal{L}_\eta \chi(y, \eta) \rho_y(y, \eta) \rho_\eta(\eta) \, dy \, d\eta = 0.$$

Proof. We first note that, provided (5.23) holds, then

$$g^{-1}(x, \mathcal{C}\eta) = g^{-1}(x^{\perp}, \eta),$$

and

$$|g|\,(x, \mathcal{C}\eta) = |g|\,(x^{\perp}, \eta).$$

Consider the equation for χ

$$-\nabla \cdot \left(\sqrt{|g|\,(y,\eta)}g^{-1}(y,\eta)\nabla\chi^e(y,\eta)\right) = H(y,\eta)\sqrt{|g|\,(y,\eta)}.$$

Making the substitution $\eta \to \mathcal{C}\eta$, then using the relations for g^{-1} and $|g|$ and changing variables in y we have

$$-\nabla \cdot \left(\sqrt{|g|\,(y,\eta)}g^{-1}(y,\eta)\nabla\tilde{\chi}(y,\eta)\right)\Big|_{y=y^{\perp}} = -H(y,\eta)\sqrt{|g|\,(y,\eta)}\Big|_{y=y^{\perp}}.$$

where $\tilde{\chi}(y,\eta) = \chi(y^{\perp}, \mathcal{C}\eta)$. It follows that

$$\chi(y^{\perp}, \mathcal{C}\eta) = -\chi(y,\eta). \tag{6.2}$$

Applying (6.2) and using the fact that Γ, Π, and $\mathcal{C}$ commute, we obtain

$$\begin{aligned}
-\mathcal{L}_\eta\chi^e(y^{\perp},\eta) &= -\Gamma\eta\cdot\mathcal{C}\nabla_\eta\chi^e(y,\mathcal{C}^\top\eta) + \mathcal{C}^\top\Gamma\Pi\mathcal{C} : \nabla\nabla\chi^e(y,\mathcal{C}^\top\eta) \\
&= -\Gamma\mathcal{C}^\top\eta\cdot\nabla_\eta\chi^e(y,\mathcal{C}^\top\eta) + \Gamma\Pi : \nabla\nabla\chi^e(y,\mathcal{C}^\top\eta) \\
&= \mathcal{L}_\eta\chi^e(y,\mathcal{C}^\top\eta).
\end{aligned}$$

Using the invariance of ρ_η with respect to $\mathcal{C}$ the effective drift term V will then be given by

$$\begin{aligned}
V &= \int_{\mathbb{R}^K}\int_{\mathbb{T}^2} \mathcal{L}_\eta\chi^e(y,\mathcal{C}\eta)\rho_y(y,\mathcal{C}\eta)\rho_\eta(\eta)\,dyd\eta \\
&= \int_{\mathbb{R}^K}\int_{\mathbb{T}^2} \mathcal{L}_\eta\chi^e(y^{\perp},\eta)\rho_y(y^{\perp},\eta)\rho_\eta(\eta)\,dyd\eta \\
&= -\int_{\mathbb{R}^K}\int_{\mathbb{T}^2} \mathcal{L}_\eta\chi^e(y,\mathcal{C}^\top\eta)\rho_y(y,\mathcal{C}^\top\eta)\rho_\eta(\eta)\,dyd\eta \\
&= -V,
\end{aligned}$$

proving the result. □

As noted in the discussion subsequent to (5.23), this will hold trivially for the fluctuating Helfrich-elastic membrane model. We first consider the case where $1 < \beta \leq \frac{3}{2}$. Since the centering condition holds for $F(y,\eta)$, by Proposition A.2.2, there exists a unique solution χ of the cell equation, given by

$$-\mathcal{L}_y\chi(y,\eta) = F(y,\eta) \qquad (y,\eta) \in \mathbb{T}^2 \times \mathbb{R}^K,$$

such that $\int_{\mathbb{T}^2} \chi(y,\eta)\rho_y(y,\eta)\,dy = 0$. By Itô's formula we have that

$$
\begin{aligned}
\epsilon^{-1}F(Y^\epsilon(t),\eta^\epsilon(t)) = \epsilon\left(\chi(Y^\epsilon(0),\eta^\epsilon(0)) - \chi(Y^\epsilon(t),\eta^\epsilon(t))\right)& \\
+ \int_0^t \sqrt{2\Sigma(Y^\epsilon(s),\eta^\epsilon(s))}\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\,dB(s)& \\
+ \epsilon^{1-\beta}\int_0^t \mathcal{L}_\eta\chi(Y^\epsilon(s),\eta^\epsilon(s))\,ds& \\
+ \epsilon^{1-\beta/2}\int_0^t \sqrt{2\Gamma\Pi}\nabla_\eta\chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\,dW(s).&
\end{aligned}
$$

We note that for $\beta > 1$, the $\epsilon^{1-\beta}$ term is singular. By introducing another auxiliary equation and using the ergodicity of $\eta^\epsilon(t)$ we are able to decompose this term into a vanishing drift term and a number of remainder terms which converge weakly in $C([0,T]\,;\mathbb{R}^2)$ to zero. To this end, let $\chi_2 : \mathbb{T}^2\times\mathbb{R}^K \to \mathbb{R}^2$ be the unique solution of

$$
\mathcal{L}_y\chi_2(y,\eta) = -\left(\mathcal{L}_\eta\chi(y,\eta) - \int_{\mathbb{T}^2}\mathcal{L}_\eta\chi(y,\eta)\rho_y(y,\eta)\,dy\right),
$$

such that $\int_{\mathbb{T}^2}\chi_2(y,\eta)\rho_y(y,\eta)\,dy = 0$. Then applying Itô's formula for $\chi_2(Y^\epsilon(t),\eta^\epsilon(t))$

$$
\begin{aligned}
\epsilon^{-1}F(Y^\epsilon(t),\eta^\epsilon(t)) = \epsilon\left(\chi(Y^\epsilon(0),\eta^\epsilon(0)) - \chi(Y^\epsilon(t),\eta^\epsilon(t))\right)& \\
+ \epsilon^{3-\beta}\left(\chi_2(Y^\epsilon(0),\eta^\epsilon(0)) - \chi_2(Y^\epsilon(t),\eta^\epsilon(t))\right)& \\
+ \int_0^t \sqrt{2\Sigma(Y^\epsilon(s),\eta^\epsilon(s))}\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\,dB(s)& \\
+ \epsilon^{3-2\beta}\int_0^t \mathcal{L}_\eta\chi_2(Y^\epsilon(s),\eta^\epsilon(s))\,ds& \\
+ \epsilon^{3-\beta/2}\int_0^t \sqrt{2\Gamma\Pi}\nabla_\eta\chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\,dW(s)& \\
+ \epsilon^{3-3\beta/2}\int_0^t \sqrt{2\Gamma\Pi}\nabla_\eta\chi_2(Y^\epsilon(s),\eta^\epsilon(s))^\top\,dW(s)& \\
+ \epsilon^{1-\beta}\int_{\mathbb{T}^2}\mathcal{L}_\eta\chi(y,\eta^\epsilon(s))\rho_y(y,\eta^\epsilon(s))\,dy&
\end{aligned}
\tag{6.3}
$$

Moreover, arguing similarly to Lemma A.1.3 one can show that

$$
\begin{aligned}
&\epsilon^{1-\beta}\int_0^t\int_{\mathbb{T}^2}\mathcal{L}_\eta\chi(y,\eta^\epsilon(s))\,\rho_y(y,\eta^\epsilon(s))\,dy\,ds \\
&\quad = \epsilon^{1-\beta}\int_{\mathbb{R}^K}\int_{\mathbb{T}^2}\mathcal{L}_\eta\chi(y,\eta)\,\rho_y(y,\eta)\rho_\eta(\eta)\,dy\,d\eta + R_1^\epsilon(t),
\end{aligned}
$$

where the remainder term $R_1^\epsilon(t) \approx O(\epsilon^{1-\beta+\beta/2}) = O(\epsilon^{1-\beta/2})$. Moreover, since condition (5.23) holds, applying Lemma 6.2.1 it follows that

$$\int_{\mathbb{R}^K} \int_{\mathbb{T}^2} \mathcal{L}_\eta \chi(y,\eta)\, \rho_y(y,\eta) \rho_\eta(\eta)\, dy\, d\eta = \mathbf{0},$$

so that

$$X^\epsilon(t) - X^\epsilon(0) = M^\epsilon(t) + R^\epsilon(t),$$

where

$$M^\epsilon(t) = \int_0^t \sqrt{2\Sigma(Y^\epsilon(s), \eta^\epsilon(s))}\left(I + \nabla_y \chi(Y^\epsilon(s), \eta^\epsilon(s))\right)^\top dB(s),$$

and $R^\epsilon(t) = O(\epsilon^{1-\beta/2})$. Therefore, arguing as in the previous case, we see that $X^\epsilon(t)$ converges weakly in $C([0,T];\mathbb{R}^2)$ to $\sqrt{2D}\, B(t)$.

For $\frac{3}{2} < \beta \leq \frac{5}{3}$, introduce the additional corrector χ_3 to be the unique solution of

$$\mathcal{L}_y \chi_3(y,\eta) = -\left(\mathcal{L}_\eta \chi_2(y,\eta) - \int_{\mathbb{T}^2} \mathcal{L}_\eta \chi_2(y,\eta) \rho_y(y,\eta)\, dy\right),$$

such that $\int_{\mathbb{T}^2} \chi_3(y,\eta)\rho_y(y,\eta)\, dy = 0$. Substituting for $\epsilon^{3-2\beta}\chi_2$ in terms of χ_3 as before using Itô's formula,

$$\begin{aligned} X^\epsilon(t) = X^\epsilon(0) &+ \int_0^t \sqrt{2\Sigma(Y^\epsilon(s), \eta^\epsilon(s))}\left(I + \nabla_y \chi(Y^\epsilon(s), \eta^\epsilon(s))\right)^\top dB(s) \\ &+ \epsilon^{3-\beta/2} \int_{\mathbb{R}^K} \int_{\mathbb{T}^2} \mathcal{L}_\eta \chi_2(y,\eta) \rho_y(y,\eta) \rho_\eta(\eta)\, dy\, dy + O(\epsilon^{3-3\beta/2}) \end{aligned}$$

Since

$$H(y,\eta) := \left(\mathcal{L}_\eta \chi(y,\eta) - \int_{\mathbb{T}^2} \mathcal{L}_\eta \chi(y,\eta) \rho_y(y,\eta)\, dy\right),$$

satisfies condition (6.1), by Lemma 6.2.1 the term

$$\int_{\mathbb{R}^K} \int_{\mathbb{T}^2} \mathcal{L}_\eta \chi_2(y,\eta) \rho_y(y,\eta) \rho_\eta(\eta)\, dy\, d\eta$$

is zero, so that $X^\epsilon(t)$ converges weakly in $C([0,T];\mathbb{R}^2)$ to $\sqrt{2D}B(t)$.

More generally, if $\frac{2k-1}{k} < \beta \leq \frac{2k+1}{k+1}$, for $k \in \mathbb{N}$, then let χ_k be the unique solution of

$$\mathcal{L}_y \chi_k(y,\eta) = -\left(\mathcal{L}_\eta \chi_{k-1}(y,\eta) - \int_{\mathbb{T}^2} \mathcal{L}_\eta \chi_{k-1}(y,\eta) \rho_y(y,\eta)\, dy\right),$$

such that $\int_{\mathbb{T}^2} \chi_k(y,\eta)\rho_y(y,\eta)\, dy = 0$. Then by continuing as before, and applying

Lemma 6.2.1 iteratively, we can show that

$$X^\epsilon(t) = X^\epsilon(0) + \int_0^t \sqrt{2\Sigma(Y^\epsilon(s),\eta^\epsilon(s))}\left(I + \nabla_y \chi(Y^\epsilon(s),\eta^\epsilon(s))\right)^\top dB(s) + O(\epsilon^{(2k+1)-(2k+1)\beta/2}),$$

so that the process $X^\epsilon(t)$ converges weakly in $C([0,T];\mathbb{R}^2)$ to $\sqrt{2D}\,B(t)$.

6.3 PROOF FOR $\alpha = 1$, $\beta > 2$

We proceed similarly to Section 3.10.5 of [Bensoussan et al., 1978]. Set $r = \beta - 2$. For $i \in \mathbb{N}$, let $\phi_i : \mathbb{T}^2 \times \mathbb{R}^K \to \mathbb{R}$ be chosen such that

$$\begin{aligned}\sum_{i=0}^{\infty} \epsilon^{ri-1} \int_0^t \mathcal{L}_y \phi_i(Y^\epsilon(s),\eta^\epsilon(s))\,ds + \sum_{i=0}^{\infty} \epsilon^{ri+1-\beta} \int_0^t \mathcal{L}_\eta \phi_i(Y^\epsilon(s),\eta^\epsilon(s))\,ds \\ = -\epsilon^{-1}\int_0^t F(Y^\epsilon(s),\eta^\epsilon(s))\,ds + O(1).\end{aligned} \tag{6.4}$$

If this condition holds then applying Itô's formula

$$\begin{aligned}\epsilon^{-1}&\int_0^t F(Y^\epsilon(s),\eta^\epsilon(s))\,ds \\ &= \sum_{i=0}^{\infty} \epsilon^{ri+1}\phi_i(Y^\epsilon(0),\eta^\epsilon(0)) - \phi_i(Y^\epsilon(t),\eta^\epsilon(t)) \\ &+ \sum_{i=0}^{\infty} \epsilon^{ri+1} \int_0^t \sqrt{2\Sigma(Y^\epsilon(s),\eta^\epsilon(s))}\nabla_y \phi_i(Y^\epsilon(s),\eta^\epsilon(s))^\top dB(s) \\ &+ \sum_{i=0}^{\infty} \epsilon^{ri+1-\beta/2} \int_0^t \sqrt{2\Gamma\Pi}\nabla_\eta \phi_i(Y^\epsilon(s),\eta^\epsilon(s))^\top dB(s) \\ &+ O(1).\end{aligned} \tag{6.5}$$

Comparing coefficients of the highest powers of ϵ in (6.4), we obtain the following relationships

$$\mathcal{L}_\eta \phi_0 = 0, \tag{6.6}$$

and

$$\mathcal{L}_\eta \phi_1 + \mathcal{L}_y \phi_0 = -F. \tag{6.7}$$

Equation (6.6) implies that ϕ_0 must be constant in η. Integrating (6.7) with respect ρ_η we have that

$$\overline{\mathcal{L}_y}\phi_0(y) = -\overline{F}(y),$$

where $\overline{\mathcal{L}_y} = \overline{F(y)} \cdot \nabla_y + \overline{\Sigma(y)} : \nabla_y \nabla_y$, for

$$\overline{F}(y) = \int_{\mathbb{R}^K} F(y,\eta)\rho_\eta(\eta)d\,\eta \qquad \text{and} \qquad \overline{\Sigma(y)} = \int_{\mathbb{R}^K} \Sigma(y,\eta)\rho_y(\eta)\,d\eta.$$

If $\beta > 3$, then the remaining terms involving ϕ_0 and ϕ_1 are of order ϵ^γ for $\gamma > 0$, and so we need not further expand the series of ϕ_i beyond $i = 1$. Moreover the functions ϕ_0 and ϕ_1 are defined uniquely by equations (6.6) and (6.7).

Since we are considering the special case of lateral diffusion on a fluctuating Helfrich surface one can obtain explicit solutions for ϕ_0 and ϕ_1. Indeed, arguing as in Theorem 5.1.3 we can show that

$$\overline{F}(y) = 0 \qquad \text{and} \qquad \overline{\Sigma}(y) = \frac{1}{2}\left(1 + \int_{\mathbb{R}^K} \frac{1}{|g|\,(y,\eta)}\rho_\eta(\eta)d\eta\right),$$

where $\Sigma(y)$ is independent of $y \in \mathbb{T}^2$. Substituting these terms in equations (6.6) and (6.7) we see that $\phi_1 = 0$ and ϕ_1 satisfies

$$-\mathcal{L}_\eta \phi_1(y,\eta) = F(y,\eta), \qquad (y,\eta) \in \mathbb{T}^2 \times \mathbb{R}^K.$$

It follows that,

$$X^\epsilon(t) = X^\epsilon(0) + M^\epsilon(t) + R^\epsilon(t), \tag{6.8}$$

where

$$M^\epsilon(t) = \int_0^t \sqrt{2\Sigma(Y^\epsilon(s), \eta^\epsilon(s))}\, dB(s),$$

and the remainder term $R^\epsilon(t)$ is $O(\epsilon^{\beta-3})$, and moreover converges weakly in $C([0,T], \mathbb{R}^2)$ to zero.

The quadratic variation of the martingale term in the above equation is given by

$$[\![M^\epsilon(t)]\!] := 2\int_0^t \Sigma(Y^\epsilon(s), \eta^\epsilon(s))\, ds.$$

Now, fix $y \in \mathbb{T}^2$, let $\chi(y,\eta)$ be the unqiue solution to the following Poisson equation

$$\mathcal{L}_\eta \chi(y,\eta) = \Sigma(y,\eta) - \overline{\Sigma},$$

such that $\int_{\mathbb{R}^K} \chi(y,\eta)\rho_\eta(\eta)\, d\eta = 0$. Then, by Itô's formula

$$\begin{aligned}
\int_0^t \chi(Y^\epsilon(s), \eta^\epsilon(s))\, ds - t\overline{\Sigma} &= \epsilon^\beta(\chi(Y^\epsilon(0), \eta^\epsilon(0)) - \chi(Y^\epsilon(t), \eta^\epsilon(t))) \\
&\quad + \epsilon^{\beta-2}\int_0^t \mathcal{L}_y \chi(Y^\epsilon(s), \eta^\epsilon(s))\, ds \\
&\quad + \epsilon^{\beta/2}\int_0^t \sqrt{2\Gamma\Pi}\nabla_\eta \chi(Y^\epsilon(s), \eta^\epsilon(s))^\top\, dW(s) \\
&\quad + \epsilon^{\beta-1}\int_0^t \sqrt{\Sigma(Y^\epsilon(s), \eta^\epsilon(s))}\nabla_y \chi(Y^\epsilon(s), \eta^\epsilon(s))^\top\, dB(s) \\
&= O(\epsilon^{\beta-2}).
\end{aligned}$$

Applying Lemma A.1.1, it follows that for each t, as $\epsilon \to 0$, the quadratic variation $[\![M^\epsilon(t)]\!]$ converges in L^2 to $t\,\overline{\Sigma}$. We then apply the Martingale central limit theorem, as before to conclude the proof.

In conclusion, for the particular case of Helfrich surface fluctuations and $\beta > 3$, the process $X^\epsilon(t)$ converges weakly in $C([0,T]; R^2)$ to the process $\sqrt{2\overline{\Sigma}}B(t)$, which is the adiabatic limit one obtains in the $(\alpha, \beta) = (0, 1)$ regime.

6.4 WHEN $2 < \beta \leq 3$

When $\beta = 3$, then the term $\int_0^t \mathcal{L}_y\phi_1(Y^\epsilon(s), \eta^\epsilon(s))\, ds$ appears in the expression for $X^\epsilon(s)$ given in (6.8). Using arguments identical to before, one can show that

$$\int_0^t \mathcal{L}_y\phi_1(Y^\epsilon(s), \eta^\epsilon(s))\, ds = \int_{\mathbb{R}^K}\int_{\mathbb{T}^2} \mathcal{L}_y\phi_1(y,\eta)\, \rho_\eta(\eta)\, dy\, d\eta + O(\epsilon).$$

It is not clear whether this term is zero or not. Therefore, we must conclude that $X^\epsilon(t)$ converges weakly in $C([0,T], \mathbb{R}^2)$ weakly to the process

$$\left[\int_{\mathbb{R}^K}\int_{\mathbb{T}^2} \mathcal{L}_y\phi_1(y,\eta)\, \rho_\eta(\eta)\, dy\, d\eta\right] t + \sqrt{2D}B(t),$$

and leave open the question whether or not the drift term is zero.

Now consider $\frac{5}{2} < \beta < 3$. For this range of β the $\mathcal{L}_y\chi_1(Y^\epsilon(s), \eta^\epsilon(s))$ term does not vanish as $\epsilon \to 0$. To this end, we introduce the function ϕ_2 which satisfies

$$\mathcal{L}_\eta\phi_2(y,\eta) = -\mathcal{L}_y\phi_1(y,\eta), \qquad (y,\eta) \in \mathbb{T}^2 \times \mathbb{R}^K.$$

More generally, for $\frac{2i+3}{i+1} < \beta \leq \frac{2i+1}{i}$, we must introduce ϕ_i into the series, to satisfy

$$\mathcal{L}_\eta\phi_i(y,\eta) = -\mathcal{L}_y\phi_{i-1}(y,\eta), \qquad (y,\eta) \in \mathbb{T}^2 \times \mathbb{R}^K.$$

The compatibility condition for the existence of ϕ_i is that

$$\int \mathcal{L}_y\phi_{i-1}(y,\eta)\rho_\eta(\eta)\, d\eta = 0,$$

however, it is not entirely clear that this condition is satisfied in even in the case of fluctuating Helfrich surfaces. Thus for the range $2 < \beta < 3$, it is still inconclusive whether or not a scaling limit exist and what the limiting equation is.

6.5 CONCLUSIONS AND FURTHER DIRECTIONS

In this chapter we have “completed the picture" for the particular model of lateral diffusion on a thermally-excited membrane. By very simply adapting the methods which are used in Appendix A we have been able to identify the distinguished limits of the system. Indeed, we have showed that, apart from a single interval, the scaling limits of the system correspond to the scaling limits for the Case I - IV regimes.

The remaining work would be to study the “gap" where $\alpha = 1$, and $2 < \beta < 3$ more carefully, and establish whether or not the auxiliary equations that arise in that regime have a solution in general, or at least conditions which would permit such solutions to exist.

Appendix A

PROOFS OF CONVERGENCE THEOREMS FOR CHAPTER 5

In this appendix we provide rigorous proofs of the three homogenization theorems of Chapter 5, namely Theorems 5.1.1, 5.2.2 and 5.3.3. We adopt a probabilistic approach throughout and directly prove the weak convergence of the stochastic differential equation (S3) to its limiting diffusion process. The corresponding PDE result will follow from the Feynmann Kac formula.

The following straightforward lemma is required in each case and is used to prove the compactness of the family of processes $\{X^\epsilon(\cdot)\}_{\epsilon>0}$.

Lemma A.0.1. *Suppose $\phi : \mathbb{T}^d \times \mathbb{R}^K \to \mathbb{R}$ satisfies*

$$|\phi(y,\eta)| \leq C(1+|\eta|^q), \qquad (y,\eta) \in \mathbb{T}^d \times \mathbb{R}^K,$$

for some positive constants C and q. Then, for $(Y^\epsilon(t), \eta^\epsilon(t))$ given in (5.10) with $\eta^\epsilon(0) \sim \rho_\eta$, where ρ_η is the invariant density of $\eta^\epsilon(t)$ given by (5.3):

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \epsilon\,|\,\phi(Y^\epsilon(t), \eta^\epsilon(t))|\right] \to 0,$$

as $\epsilon \to 0$

Proof. Let $V(\eta) = (1+|\eta|^q)$. Applying Itô's formula to the function V, for $0 \leq t \leq T$,

$$\begin{aligned} V(\eta^\epsilon(t)) = V(\eta(0)) &- \int_0^t \Gamma\nabla_\eta V(\eta^\epsilon(s))\,ds \\ &+ \int_0^t \Gamma\Pi : \left(\nabla_\eta\nabla_\eta V(\eta^\epsilon(s))\right)\,ds \\ &+ \int_0^t \sqrt{2\Gamma\Pi}\nabla_\eta V(\eta^\epsilon(s))\,dW(s). \end{aligned}$$

Using the fact that $\eta^\epsilon(t)$ is a stationary Gaussian process which has finite moments of all orders, bounded uniformly of ϵ, and applying the Burkholder-Davis-Gundy inequality for the final term, we see immediately that

$$\mathbb{E} \sup_{0\leq t\leq T} |\phi(Y^\epsilon(t), \eta^\epsilon(t))| \leq K_1 + K_2 T + K_3\sqrt{T},$$

for some constants K_1, K_2 and K_3 which depend on Γ and Π and q. Since the RHS is bounded independently of ϵ the result follows. □

A.1 CASE II

It should be noted that Theorem 5.1.1 follows directly from Theorem 3 of [Pardoux and Veretennikov, 2001], however since the analysis is considerably simplified by the fact that the fast process $\eta^\epsilon(t)$ is an Ornstein-Uhlenbeck process and since we assume that the coefficients of $X^\epsilon(t)$ are smooth, we are able provide a simpler, direct proof instead. The approach to proving the limit theorem is similar to that of Section 3.11 of [Bensoussan et al., 1978] in that we show that any limiting process must solve a particular martingale problem corresponding to a weak solution of the SDE (5.6).

The following lemma guarantees existence of a smooth solution ϕ to the Poisson equation (A.2) and provides $L^2(\rho_\eta)$ bounds on ϕ and its first and second derivatives.

Lemma A.1.1. *Let $b : \mathbb{R}^d \times \mathbb{R}^K \to \mathbb{R}$, be a smooth function such that*

$$|b(x,\eta)| + |\nabla_x b(x,\eta)| + |\nabla_x\nabla_x b(x,\eta)| < C(1+|\eta|^q), \qquad (x,\eta) \in \mathbb{R}^d \times \mathbb{R}^K, \quad \text{(A.1)}$$

for some positive constants C and q independent of x. Suppose that for each fixed x, $b(x,\cdot)$ is centered with respect to ρ_η. Then there exists a unique solution ϕ to the equation

$$\mathcal{L}_0\phi(x,\eta) = -b(x,\eta), \qquad (x,\eta) \in \mathbb{R}^d \times \mathbb{R}^K, \quad \text{(A.2)}$$

satisfying

$$\int \phi(x,\eta)\rho_\eta(d\eta) = 0,$$

and $\phi(x,\cdot)$ for each $x \in \mathbb{R}^d$. Moreover, $\phi(x,\cdot)$ is smooth in η and

$$\|\phi(x,\cdot)\|_{L_2(\rho_\eta)} + \|\nabla_x\phi(x,\cdot)\|_{L_2(\rho_\eta)} + \|\nabla_x\nabla_x\phi(x,\cdot)\|_{L_2(\rho_\eta)} < C'$$

for some constant C', independent of x.

Proof. Let $P(t)$ be the Markov semigroup corresponding to the OU process $\eta(t)$ with invariant measure ρ_η. From [Metafune et al., 2002] we know that the infinitesimal generator of $P(t)$ is $\mathcal{L}_0$ given by (5.2) with domain $D(\mathcal{L}_0)$ equal to the weighted Sobolev space $W^{2,2}(\mathbb{R}^K;\rho_\eta)$, and moreover the process has an $L^2(\rho_\eta)$ spectral gap, so that, for some positive μ,

$$\|P(t)b(x,\cdot)\|_{L^2(\rho_\eta)} \leq Ce^{-\mu t} \|b(x,\cdot)\|_{L^2(\rho_\eta)}, \qquad t \geq 0 \tag{A.3}$$

for some positive C. Define

$$\phi(x,\eta) = \int_0^\infty P(t)b(x,\eta)dt.$$

Then by (A.3) it follows that $\|\phi(x,\cdot)\|_{L_2(\rho_\eta)} < \frac{C}{\mu} \|b(x,\cdot)\|_{L_2(\rho_\eta)} =: C_0'$. Moreover, using the identity,

$$P(t)\phi - \phi = \int_0^t P(s)b\, ds,$$

we see that $\lim_{t\to 0}(P(t)\phi - \phi)$ converges strongly in $L^2(\rho_\eta)$ so that $\phi \in \mathcal{D}(\mathcal{L}_0)$ and ϕ solves (A.2) in the sense of distributions. Since $\mathcal{L}_0$ satisfies the conditions of Hörmander's theorem, [Rogers and Williams, 2000, Theorem 38.16], and $b(x,\cdot)$ is smooth, it follows that $\phi(x,\cdot)$ is also smooth. Since we have the bounds (A.1) for the derivatives of $b(x,\eta)$ with respect to x, it follows by the dominated convergence theorem that

$$(\nabla_x)^k P(t)b(x,\eta) = P(t)(\nabla_x)^k b(x,\eta),$$

for $k = 1, 2$, from which the rest of the lemma follows. □

Let $X^\epsilon(t)$ be the "slow" process corresponding to the particle trajectory, given by (5.1). The following lemma shows that the family $\{X^\epsilon(\cdot)\}_{\epsilon>0}$ is tight, so that the family contains a limit point in the topology of weak convergence on $C([0,T];\mathbb{R}^d)$

Lemma A.1.2. *Suppose the $X^\epsilon(\cdot)$ with paths in $C([0,T];\mathbb{R}^d)$ has initial condition $X^\epsilon(0) = x$ for all $\epsilon > 0$. Then $\{X^\epsilon\}_{\epsilon>0}$ is tight.*

Proof. We simply verify conditions (i) and (ii) specified in Theorem 8.3 of Billingsley [Billingsley, 2009]. Condition (i) holds trivially. What remains is to prove Condition (ii), namely that for any $\delta > 0$, there exists ϵ_0 and $\gamma > 0$ such that

$$\gamma^{-1} \sup_{0<\epsilon<\epsilon_0} \sup_{0\leq t_0 \leq T} \mathbb{P}\left(\sup_{t\in[t_0,t_0+\gamma]} |X^\epsilon(t) - X^\epsilon(t_0)| \geq \delta \right) \leq \delta.$$

Following [Pardoux and Veretennikov, 2001], this condition will follow from the following two estimates. Let $\nu > 0$, then

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\le t\le t_0+\gamma}\left|\int_{t_0}^{t} F(X^\epsilon(t),\eta^\epsilon(t))\,ds\right|^{1+\nu}\right)\\
&\le C\gamma^\nu \mathbb{E}\int_{t_0}^{t_0+\gamma} |F(X^\epsilon(t),\eta^\epsilon(t))|^{1+\nu}\,ds\\
&\le C'\gamma^\nu \int_{t_0}^{t_0+\nu} (1+\mathbb{E}\,|\eta|^{\beta_1(1+\nu)})\,ds\\
&\le K\gamma^{1+\nu},
\end{aligned}
$$

where we used Jensen's inequality, the bounds (A.1) and the fact that the Gaussian process η has bounded moments. Similarly

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\le t\le t_0+\gamma}\left|\int_{t_0}^{t} \Sigma(X^\epsilon(t),\eta^\epsilon(t))\,dB(s)\right|^{2+2\nu}\right)\\
&\le \mathbb{E}\left(\int_{t_0}^{t_0+\gamma} |\Sigma(X^\epsilon(t),\eta^\epsilon(t))|_F^2\,ds\right)^{1+\nu}\\
&\le \gamma^{1+\nu},
\end{aligned}
$$

where we applied the Burkholder-Gundy-Davis inequality in the second line and the fact that Σ is bounded in the third line. □

It follows from Prokhorov's theorem that the family $\{X^\epsilon\}_{\epsilon>0}$ is relatively compact in the Skorohod topology of weak convergence of stochastic processes taking paths in $C([0,T];\mathbb{R}^d)$. In particular, there exists a process X^0 whose paths lie in $C([0,T];\mathbb{R}^d)$ such that $X^{\epsilon_n} \Rightarrow X^0$ along a subsequence ϵ_n. We now show that any limit point of the family $\{X^\epsilon\}_{\epsilon>0}$ must be a weak solution of the SDE (5.6), thus proving Theorem 5.1.1.

The following lemma will be required to identify the limiting process X^0 of the family $\{X^\epsilon\}_{\epsilon>0}$ in the proofs of both Proposition 5.1.1 and Theorem 5.2.2.

Lemma A.1.3. *Suppose $b : \mathbb{R}^d \times \mathbb{R}^K \to \mathbb{R}$ satisfies the conditions of Lemma A.1.1 and suppose $f \in C_b^2(\mathbb{R}^d;\mathbb{R})$. Then*

$$
\mathbb{E}\left|\int_{t_0}^{t} b(X^\epsilon(s),\eta^\epsilon(s))f(X^\epsilon(s))\,ds\,\middle|\mathcal{F}_{t_0}\right|^2 \to 0, \tag{A.4}
$$

as $\epsilon \to 0$ for $0 \le t_0 \le t \le T$. Moreover the process defined by

$$
t \to \int_0^t b(X^\epsilon(s),\eta^\epsilon(s))f(X^\epsilon(s))\,ds \tag{A.5}
$$

converges weakly in $C([0,T];\mathbb{R}^d)$ *to* 0 *as* $\epsilon \to 0$.

Proof. By Lemma A.1.1 there exists a unique function ϕ which solves

$$\mathcal{L}_0\phi(x,\eta) = -b(x,\eta), \qquad \text{for } (x,\eta) \in \mathbb{R}^d \times \mathbb{R}^K.$$

Since ϕ is sufficiently smooth we can apply Itô's formula directly to see that

$$\begin{aligned}\phi(X^\epsilon(t),\eta^\epsilon(t))f(X^\epsilon(t)) - \phi(X^\epsilon(t_0),\eta^\epsilon(t_0))f(X^\epsilon(0))&\\ = \int_{t_0}^t F(X^\epsilon(t),\eta^\epsilon(t))\cdot\nabla_x(\phi f)\,ds&\\ + \int_{t_0}^t \Sigma(X^\epsilon(t),\eta^\epsilon(t)) : \nabla_x\nabla_x(\phi f)\,ds&\\ + \frac{1}{\epsilon}\int_{t_0}^t \mathcal{L}_0\phi f\,ds&\\ + M_1^\epsilon(t) + \frac{1}{\sqrt{\epsilon}}M_2^\epsilon(t),&\end{aligned}$$

where $M_1^\epsilon(t)$ and $M_2^\epsilon(t)$ are L^2-martingales. It follows from the Itô isometry that

$$\begin{aligned}&\mathbb{E}\Big[\int_{t_0}^t b(X^\epsilon(s),\eta^\epsilon(s))f(X^\epsilon(s))\Big|\,\mathcal{F}_{t_0}\Big]^2\\ &\quad\le 2\epsilon^2\,\mathbb{E}\Big[|\phi(X^\epsilon(t_0),\eta^\epsilon(t_0))f(X^\epsilon(t_0))|^2 + |\phi(X^\epsilon(t),\eta^\epsilon(t))f(X^\epsilon(t))|^2\,\Big|\,\mathcal{F}_{t_0}\Big]\\ &\quad+ 2\,\epsilon^2\int_{t_0}^t \mathbb{E}\Big[|F(X^\epsilon(s),\eta^\epsilon(s))\cdot\nabla_x(\phi f)|^2 + |\Sigma(X^\epsilon(s),\eta^\epsilon(s)) : \nabla_x\nabla_x(\phi f)|^2\,\Big|\,\mathcal{F}_{t_0}\Big]\,ds\\ &\quad+ 4\,\epsilon^2\int_{t_0}^t \mathbb{E}\Big[\nabla_x(\phi f)\cdot\Sigma(X^\epsilon(s),\eta^\epsilon(s))\nabla_x(\phi f)\,ds\Big|\,\mathcal{F}_{t_0}\Big]\,ds\\ &\quad+ 4\,\epsilon\int_{t_0}^t \mathbb{E}\Big[\nabla_\eta(\phi f)\cdot\Gamma\Pi\nabla_\eta(\phi f)\,ds\Big|\,\mathcal{F}_{t_0}\Big]\,ds.\end{aligned}$$

Using the fact that ϕ and its derivatives are in $L^2(\rho_\eta)$ and that f and its derivatives are bounded, (A.4) follows. In particular, this implies that any finite dimensional distribution of the process (A.5) converges to 0. It remains to show that the family of processes is tight by applying Theorem 8.2 of [Billingsley, 2009].

Condition (i) follows trivially. To verify Condition (ii), we note that

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{|t-t_0|<\delta}\left|\int_{t_0}^{t} b(X^\epsilon(s),\eta^\epsilon(s))f(X^\epsilon(s))\,ds\right|\right)\\
&\leq \epsilon\mathbb{E}\left(\sup_{|t-t_0|<\delta}|\phi(X^\epsilon(t),\eta^\epsilon(t))f(X^\epsilon(t))-\phi(X^\epsilon(t_0),\eta^\epsilon(t_0))f(X^\epsilon(t_0))|\right)\\
&+\epsilon\,\mathbb{E}\left(\sup_{|t-t_0|<\delta}\int_{t_0}^{t}|F(X^\epsilon(t),\eta^\epsilon(t))\cdot\nabla_x(\phi f)|\,ds\right)\\
&+\epsilon\,\mathbb{E}\left(\sup_{|t-t_0|<\delta}\int_{t_0}^{t}|\Sigma(X^\epsilon(t),\eta^\epsilon(t)):\nabla_x\nabla_x(\phi f)|\,ds\right)\\
&+\epsilon\,\mathbb{E}\left(\sup_{|t-t_0|<\delta}\left|\int_{t_0}^{t}\sqrt{2\Sigma(X^\epsilon(t),\eta^\epsilon(t))}\nabla_x(\phi f)\,dB(s)\right|\right)\\
&+\sqrt{\epsilon}\,\mathbb{E}\left(\sup_{|t-t_0|<\delta}\left|\int_{t_0}^{t}\sqrt{2\Gamma\Pi}\nabla_\eta(\phi f)\,dW(s)\right|\right).
\end{aligned}
$$

Condition (ii) then follows from applying Lemma A.0.1 for the first term, the bounds on ϕ and the Burkholder-Gundy-Davis inequality for the martingale terms. □

Proof of Theorem 5.1.1. Let $f\in C_b^2(\mathbb{R}^d)$ and fix $t_0>0$. By Itô's formula

$$
\begin{aligned}
\mathbb{E}\Big[&f(X^\epsilon(t))-f(X^\epsilon(t_0))\\
&-\int_{t_0}^{t}F(X^\epsilon(s),\eta^\epsilon(s))\cdot\nabla_x f\,ds\\
&-\int_{t_0}^{t}\Sigma(X^\epsilon(s),\eta^\epsilon(s)):\nabla_x\nabla_x f(x)\,ds\quad\Big|\mathcal{F}_{t_0}\Big]=0.
\end{aligned}
$$

Let $\overline{F}$ and $\overline{\Sigma}$ be as in (5.7) and (5.8) respectively, then

$$
\begin{aligned}
\mathbb{E}\Big[&f(X^\epsilon(t))-f(X^\epsilon(t_0))\\
&-\int_{t_0}^{t}\overline{F}(X^\epsilon(s))\cdot\nabla_x f\,ds\\
&-\int_{t_0}^{t}\overline{\Sigma}(X^\epsilon(s)):\nabla_x\nabla_x f\,ds\\
&-\int_{t_0}^{t}b_1(X^\epsilon(s),\eta^\epsilon(s))\cdot\nabla_x f\,ds\\
&-\int_{t_0}^{t}b_2(X^\epsilon(s),\eta^\epsilon(s)):\nabla_x\nabla_x f\,ds\Big|\mathcal{F}_{t_0}\Big]=0,
\end{aligned}
$$

where

$$b_1(x,\eta) = \left(F(x,\eta) - \overline{F}(x)\right),$$

and

$$b_2(x,\eta) = \left(\Sigma(x,\eta) - \overline{\Sigma}(x)\right).$$

Taking the weak limit of X^ϵ along the subsequence ϵ_n,

$$\begin{aligned}
&\mathbb{E}\Big[f(X^0(t)) - f(X^0(t_0)) \\
&\qquad - \int_{t_0}^{t} \overline{F}(X^0(s)) \cdot \nabla_x f\, ds \\
&\qquad - \int_{t_0}^{t} \overline{\Sigma}(X^0(s)) : \nabla_x \nabla_x f\, ds \Big| \mathcal{F}_{t_0}\Big] \\
&= \lim_{\epsilon_n \to 0} \mathbb{E}\Big[\int_{t_0}^{t} b_1(X^\epsilon(s), \eta^\epsilon(s)) \cdot \nabla_x f\, ds + \int_{t_0}^{t} b_2(X^\epsilon(s), \eta^\epsilon(s)) : \nabla_x \nabla_x f\, ds \Big| \mathcal{F}_{t_0}\Big].
\end{aligned}$$

However, the two terms on the RHS are zero by an application of Lemma A.1.3. It follows that any limit point X^0 of $\{X^\epsilon\}_{\epsilon>0}$ must satisfy the following relation for any $0 \leq t_0 \leq T$.

$$\begin{aligned}
&\mathbb{E}\Big[f(X^0(t)) - f(X^0(t_0)) \\
&\qquad - \int_{t_0}^{t} \overline{F}(X^0(s)) \cdot \nabla_x f\, ds \\
&\qquad - \int_{t_0}^{t} \overline{\Sigma}(X^0(s)) : \nabla_x \nabla_x f\, ds \Big| \mathcal{F}_{t_0}\Big] = 0.
\end{aligned}$$

Since we are considering a process with taking continuous paths, this implies that X^0 is a solution to the martingale problem for $\mathcal{L}_0$ given by

$$\mathcal{L}_0 = \overline{F}(x) \cdot \nabla_x + \overline{\Sigma}(x) : \nabla_x \nabla_x.$$

Since $\overline{F}$ is bounded and $\overline{\Sigma}$ is continuous, bounded and strictly positive definite, by the Stroock-Varadhan theorem [Rogers and Williams, 2000, Theorem 24.1], the martingale problem for $\mathcal{L}_0$ possesses a unique solution. Therefore X^0 is the unique (in the weak sense) limit point of the family $\{X^\epsilon\}_{\epsilon>0}$. Moreover, by Theorem 20.1 of [Rogers and Williams, 2000], the process X^0 will be the unique weak solution of the SDE (5.6), completing the proof.

A.2 CASE III

In this section we derive a proof of the homogenization result for the $(\alpha, \beta) = (1, 1)$ scaling. The probabilistic approach which we adopt is similar to that of [Garnier, 1997] who considers homogenization of SDEs with time-dependent coefficients. Our case is slightly more complicated in that the temporal fluctuations are Gaussian and therefore unbounded, however, we see that the underlying structure of the proof is nearly identical.

The approach resembles that of a reiterated homogenization problem. Since the spatial fluctuations $Y^\epsilon(t)$ relax to equilibrium at a faster scale than the temporal fluctuations $\eta^\epsilon(t)$, we can view $\eta^\epsilon(t)$ as "frozen" and do standard homogenization with respect to the fast process $Y^\epsilon(t)$.

The proof contains two basic steps. Firstly, using the Itô's formula, we decompose the slow process $X^\epsilon(t)$ given in (5.10) into a martingale $M^\epsilon(t)$, a remainder term $R^\epsilon(t)$ which vanishes asymptotically as $\epsilon \to 0$ and an $O(1)$ drift term $L\,t$, with constant coefficient L given by (2.29). Each of these terms can be written in terms of the solution of a Poisson equation of the form

$$\begin{aligned} -\mathcal{L}_0\phi(y,\eta) &= F(y,\eta), \qquad (y,\eta) \in \mathbb{T}^d \times \mathbb{R}^K, \\ \int \phi(y,\eta)\rho_y(dy,\eta) &= 0. \end{aligned} \tag{A.6}$$

where $\mathcal{L}_0$ is given by (5.11), and where η is the value of the "frozen" OU process.

Using the ergodicity of the fast process, the quadratic variation of $M^\epsilon(t)$ converges to Dt, where D is given by (5.18). and so, by the martingale central limit theorem [Ethier and Kurtz, 2009, Theorem 7.1.4], the martingale $M^\epsilon(t)$ converges weakly in $C([0,T],\mathbb{R}^d)$ to a Brownian motion with diffusion coefficient D.

The second part of the proof involves proving that the remainder term $R^\epsilon(t)$ converges weakly in $C([0,T];\mathbb{R}^d)$ to 0. Proving the convergence of the finite dimensional distributions to 0 is straightforward. What remains is to prove the tightness of the family $\{R^\epsilon(\cdot)\}_{\epsilon>0}$, after which the result will follow from Prokhorov's theorem [Ethier and Kurtz, 2009]. We use the standard tightness criteria of [Billingsley, 2009, Theorem 8.2]. Verifying these criteria require bounds on the solution of the Poisson equation (A.6) which are at most polynomially growing with respect to the "frozen" value η. To obtain these bounds we derive a spectral gap estimate depending on η, and then Use standard heat-kernel estimates such as those in [Davies, 1990, Chapter 2.4] to derive explicit bounds on ϕ.

For fixed $\eta \in \mathbb{R}^K$, denote by $L^p(\rho_y(\cdot,\eta))$ the Banach space of functions $f : \mathbb{T}^d \to \mathbb{R}$ which are p-integrable with respect to $\rho_y(\cdot,\eta)$. Then the operator $\mathcal{L}_0$ defined by (5.11) is symmetric with respect to the inner product $\langle\cdot,\cdot\rangle_{\rho_y(\cdot,\eta)}$. Pointwise bounds for χ will then follow from the following two results: firstly we show that $P(t)$ possesses an $L^2(\rho_y(\cdot,\eta))$ spectral gap $\lambda(\eta)$. Secondly, we prove that the semigroup $P(t)$ is ultracontractive, i.e. that $\|P(t)\|_{2\to\infty} \le C(\eta)$, where $C(\eta)$ is growing at most polynomially in η. The equivalence between this Nash inequality and ultracontractivity of the Markov semigroup $P(t)$ has been discussed by many authors (see [Davies, 1990, Chapter 2.4], [Stroock, 1988] or [Bakry et al., 2010]), we will apply the version of the equivalence described in Theorem 2.1 of [Bakry et al., 2010].

The existence of an $L^2(\rho_y(\cdot,\eta))$-spectral gap is a a straightforward consequence of the Poincaré inequality on the domain $\mathbb{T}^d$ with respect to the Lebesgue measure.

Lemma A.2.1. *For $f \in H^1(\rho_\eta(\cdot,\eta)))$ we have that*

$$\int_{\mathbb{T}^d} f(y)^2 \rho_y(y,\eta))\,dy - \left(\int_{\mathbb{T}^d} f(y)\rho_y(y,\eta)\,dy\right)^2 \leq C(\eta)\langle(-\mathcal{L}_0)\,f, f\rangle_{\rho_y(\cdot,\eta)}$$

where

$$C(\eta) = \frac{C'}{Z(\eta)\sqrt{|g|_\infty(\eta)}},$$

where $|g|_\infty(\eta) := \sup\limits_{y\in\mathbb{T}^d} |g|(y,\eta)$ and C' and C'' are postive constants, independent of y,η. As a consequence we obtain the following spectral gap inequality

$$\|P(t)f\|_{L^2(\rho_y(\cdot,\eta))} \leq e^{-\lambda(\eta)t}\|f\|_{L^2(\rho_y(\cdot,\eta))}, \qquad t\geq 0,$$

where the spectral gap is given by $\lambda(\eta) = C(\eta)$. Note that $\lambda^{-1}(\eta) \leq C''(1+|\eta|^2)$, where C'' is a constant independent of η.

Proof. The argument here is identical to the proof of estimate (9.13) of [Komorowski et al., 2012]. We first note that

$$\begin{aligned}
&\int_{\mathbb{T}^d} f(y)^2\rho_y(y,\eta))\,dy - \left(\int_{\mathbb{T}^d} f(y)\rho_y(y,\eta)\,dy\right)^2 \\
&\quad= \frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} (f(x)-f(y))^2\rho_y(x,\eta)\rho_y(y,\eta)\,dx\,dy \\
&\quad= \frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\left[\int_0^1 t^{1/4}t^{-1/4}\nabla f(tx+(1-t)y)\cdot(x-y)\,dt\right]^2\rho_y(x,\eta)\rho_y(y,\eta)\,dx\,dy \\
&\quad\leq \frac{1}{2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\left(\int_0^1\frac{1}{\sqrt{t}}\,dt\int_0^1\sqrt{t}\,|\nabla f(tx+(1-t)y)\cdot(x-y)|^2\,dt\right)\rho_y(x,\eta)\rho_y(y,\eta)\,dx\,dy.
\end{aligned}$$

Making the change of variables $x' = tx+(1-t)y$, $y' = y$, and noting that $|x-y|^2 \leq 2$, we can write the above integral as

$$2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\int_0^1\frac{1}{\sqrt{t}}\left|\nabla f(x')\right|^2\rho_y((x'-(1-t)y')/t,\eta)\rho_y(y',\eta)\,dt\,dx'\,dy'.$$

so that, substituting the definition of $\rho_y(y,\eta)$:

$$\begin{aligned}
\int_{\mathbb{T}^d} f(y)^2 \rho_y(y,\eta))\,dy - \left(\int_{\mathbb{T}^d} f(y)\rho_y(y,\eta)\,dy\right)^2 \\
&\leq \frac{4}{Z(\eta)^2\sqrt{|g|_\infty(\eta)}}\int_{\mathbb{T}^d}\frac{|\nabla f(y)|^2}{\sqrt{|g|(y,\eta)}}\,dy \\
&\leq \frac{4}{Z(\eta)\sqrt{|g|_\infty(\eta)}}\int_{\mathbb{T}^d}\nabla f(y)\cdot g^{-1}(y,\eta)\nabla f(y)\rho_y(y,\eta)\,dy \\
&= C(\eta)\langle(-\mathcal{L}_0)\,f,f\rangle_{\rho_y(\cdot,\eta)},
\end{aligned}$$

as required □

In the following proposition we state sufficient conditions for the existence of a unique solution to the Poisson equation (A.6), and applying the spectral gap estimates of the previous proposition provide explicit pointwise bounds on the solution ϕ.

Proposition A.2.2. *Suppose $d < 4$ and $F : \mathbb{T}^d \times \mathbb{R}^K \to \mathbb{R}$ is a function such that for some $M \in \mathbb{N}$:*

1. *$F(y,\cdot) \in C^M(\mathbb{R}^K)$ and, for each $m \leq M$, each component of $\left(\nabla_\eta\right)^m F(y,\eta)$ is Hölder continuous with respect to y,*
2. *$\int F(y,\eta)\rho_y(y,\eta)\,dy = 0$, for each fixed $\eta \in \mathbb{R}^K$, and*
3. *For each $m \leq M$, there exists positive constants q_m and C_m such that*

$$\left|\left(\nabla_\eta\right)^m F(y,\eta)\right| \leq C_m(1+|\eta|^{q_m}), \qquad (y,\eta)\in\mathbb{T}^d\times\mathbb{R}^K.$$

Then there exists a unique solution $\phi : \mathbb{T}^d \times \mathbb{R}^K \to \mathbb{R}$ such that solves (A.6) which satisfies $\phi \in C^{2\times M}(\mathbb{T}^d \times \mathbb{R}^K)$. Moreover there exist constants C'_m and $q'_m > 0$ independent of y and η such that

$$\left|(\nabla_\eta)^m\phi(y,\eta)\right| \leq C'_m(1+|\eta|^{q'_m}), \qquad (y,\eta)\in\mathbb{T}^d\times\mathbb{R}^K. \tag{A.7}$$

Finally,

$$\begin{aligned}
\int\int \nabla_y\phi(y,\eta)\cdot g^{-1}(y,\eta)\nabla_y\phi(y,\eta)\rho_y(dy,\eta)\rho_\eta(d\eta) \\
= \int_{\mathbb{R}^K}\langle(-\mathcal{L}_0)\phi,\phi\rangle_{\rho_y(\cdot,\eta)}\rho_\eta(d\eta) < \infty.
\end{aligned} \tag{A.8}$$

Proof. The existence of a unique solution $\phi(\cdot,\eta)$ solving (A.6) for fixed η, and such that $\phi(\cdot,\eta)\in C^2(\mathbb{T}^d)$ follows from the Fredholm alternative, say Theorem 6.15 of [Gilbarg and Trudinger, 2001]. The fact that $\phi(y,\eta)$ has mixed continuous derivatives with respect to η up to order M then follows from a bootstrap argument using the fact that the coefficients of $\mathcal{L}_0$ and its derivatives grow at most polynomially with respect to η. The bounds in (A.8) then follow from the Cauchy-Schwartz inequality.

We now prove (A.7) for $m = 0$, noting that the bounds for higher derivatives of η will follow identically. Applying the Nash inequality for $\mathbb{T}^d$ given in Theorem B.0.2 and using the bounds on $\rho_y(y, \eta)$:

$$\|u\|_{L^2(\rho_y(\cdot,\eta))}^{1+d/2} \leq C\left(1 + |\eta|^q\right) \|u\|_{L^1(\rho_y(\cdot,\eta))} \left(\|u\|^2_{L^2(\rho_y(\cdot,\eta))} + \langle(-\mathcal{L}_0)\, u, u\rangle_{\rho_y(\cdot,\eta)}\right)^{d/4}, \tag{A.9}$$

for $q = \frac{1}{2} + \frac{1}{d}$, where C is a constant depending only on d.

Since (A.9) holds, we can apply Theorem 2.1 of [Bakry et al., 2010] which implies that

$$\|P(t)f\|_{L^2(\rho_y(\cdot,\eta))} \leq C(t) \, \|f\|_{L^1(\rho_y(\cdot,\eta))}\,,$$

where

$$C(t) = C(1 + |\eta|^{1+2/d}) \max(1, t^{-d/4})$$

for some positive constant C. Thus $P(t)$ is bounded from $L^1(\rho_y(\cdot,\eta))$ to $L^2(\rho_y(\cdot,\eta))$ with norm C(t). Applying a duality argument and using the symmetry of $P(t)$ it follows that $P(t)$ is also bounded from $L^2(\rho_y(\cdot,\eta))$ to $L^\infty(\rho_y(\cdot,\eta))$ also with norm $C(t)$, so that

$$\|P(t)f\|_{L^\infty(\mathbb{T}^d)} \leq C(1 + |\eta|^{2+2/d}) \max(1, t^{-d/4}) \, \|f\|_{L^2(\mathbb{T}^d)}\,, \tag{A.10}$$

Writing $\phi(y, \eta) = \int_0^\infty P(t)f(y, \eta)\, dt$, we split the integral into two parts:

$$\phi(y, \eta) = \int_0^1 P(t)f(y, \eta)\, dt + \int_1^\infty P(t)f(y, \eta)\, dt.$$

For the second integral we apply the L^2- spectral gap estimate to get

$$\begin{aligned}
\left|\int_1^\infty P(t)f(y, \eta)dt\right| &\leq \int_1^\infty \|P_1\|_{2\to\infty} \|P_{t-1}f\|_{L^2(\mathbb{T}^d)} \\
&\leq C(1 + |\eta|^{2+2/d}) \int_1^\infty \|P_{t-1}f\|_{L^2(\mathbb{T}^d)} \\
&\leq C(1 + |\eta|^{2+2/d}) \int_1^\infty e^{-\lambda(\eta)(t-1)} \, \|f\|_{L^2(\mathbb{T}^d)} \\
&\leq C(1 + |\eta|^{2+2/d})\lambda(\eta)^{-1} \, \|f\|_{L^2(\rho_y(\cdot,\eta))} \\
&\leq C(1 + |\eta|^{3+2/d})\lambda(\eta)^{-1} \, \|f\|_{L^2(\mathbb{T}^d)} \\
&\leq C(1 + |\eta|^{5+2/d+q_0}),
\end{aligned} \tag{A.11}$$

where the constant C in each line may differ. For the first integral we use estimate

(A.10) directly to get

$$\begin{aligned}\left|\int_0^1 P(t)f(y,\eta)\,dt\right| &\leq C(1+|\eta|^{2+2/d+q_0})\int_0^1 t^{-d/4}dt \\ &\leq C(1+|\eta|^{2+2/d+q_0}).\end{aligned} \tag{A.12}$$

Combining (A.11) and (A.12) we get the desired result with $q_0' = 5 + 2/d + q_0$. $\square$

The following lemma is is the heart of the homogenisation result. Indeed, almost every proof of the homogenization of a stochastic process will contain a lemma of this form. It is the direct analogue of Lemma A.1.3 and states the intuitively obvious result that if a sufficiently smooth function $b(y,\eta)$ has expectation 0 with respect to the invariant measure of the fast process, then as $\epsilon \to 0$, the process $\int_0^t b(Y^\epsilon(s), \eta^\epsilon(s))\,ds$ will converge to 0 weakly in $C([0,T]\,;\,\mathbb{R}^d)$.

Lemma A.2.3. *Suppose the conditions of Proposition A.2.2 are satisfied for* $b : \mathbb{T}^d \times \mathbb{R}^K \to \mathbb{R}$*, then*

$$\mathbb{E}\left|\int_{t_0}^t b(Y^\epsilon(s), \eta^\epsilon(s))\,ds\,\middle|\mathcal{F}_{t_0}\right|^2 \to 0,$$

as $\epsilon \to 0$ *for* $0 \leq t_0 \leq t \leq T$*. Moreover the process defined by*

$$t \to \int_0^t b(Y^\epsilon(s), \eta^\epsilon(s))\,ds$$

converges weakly in $C([0,T];\mathbb{R}^d)$ *to* 0.

Proof. The proof of this lemma follows almost step-by step that of the analogous result for Case II, given in Lemma A.1.3. $\square$

Using the above results, we can now provide a proof of Theorem 5.2.2, based on that of [Garnier, 1997].

Proof of Theorem 5.2.2. Since $F(y,\eta)$ is smooth and satisfies the conditions of Proposition A.2.2 with $M \geq 4$, the cell equation

$$\mathcal{L}_0\chi(y,\eta) = -F(y,\eta) \qquad (y,\eta) \in \mathbb{T}^d \times \mathbb{R}^K,$$

possesses a unique solution satisfying $\int \chi(y,\eta)\rho_y(dy,\eta) = 0$ such that $\chi \in C^{2\times 4}(\mathbb{T}^d \times \mathbb{R}^K;\mathbb{R}^d)$ and satisfies the bounds in (A.7) and (A.8). Since χ is sufficiently smooth we are justified in applying Itô's formula to $\chi(Y^\epsilon(t), \eta^\epsilon(t))$ to obtain

$$\begin{aligned}
\chi(Y^\epsilon(t),\eta^\epsilon(t)) = \chi(Y^\epsilon(0),\eta^\epsilon(0)) &+ \frac{1}{\epsilon^2}\int_0^t \mathcal{L}_0\chi(Y^\epsilon(s),\eta^\epsilon(s))\,ds \\
&+ \frac{1}{\epsilon}\int_0^t \mathcal{L}_\eta\chi(Y^\epsilon(s),\eta^\epsilon(s))\,ds \\
&+ \frac{1}{\epsilon}\int_0^t \sqrt{2\Sigma}\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\,dB(s) \\
&+ \frac{1}{\sqrt{\epsilon}}\int_0^t \sqrt{2\Gamma\Pi}\nabla_\eta\chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\,dW(s).
\end{aligned}$$

Therefore, the slow process $X^\epsilon(t)$ can be written as

$$\begin{aligned}
X^\epsilon(t) - X^\epsilon(0) = \epsilon\left(\chi(Y^\epsilon(0),\eta^\epsilon(0)) - \chi(Y^\epsilon(t),\eta^\epsilon(t))\right)& \\
+ \sqrt{\epsilon}\int_0^t \sqrt{2\Gamma\Pi}\nabla_\eta\chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\,dW(s)& \\
+ \int_0^t \mathcal{L}_\eta\chi(Y^\epsilon(s),\eta^\epsilon(s))\,ds& \\
+ \int_0^t \sqrt{2\Sigma}\left(I + \nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s))\right)^\top\,dB(s).&
\end{aligned} \tag{A.13}$$

Let $Q^\epsilon(t)$ be the process defined by

$$Q^\epsilon(t) = \epsilon\left(\chi(Y^\epsilon(0),\eta^\epsilon(0)) - \chi(Y^\epsilon(t),\eta^\epsilon(t))\right).$$

Then, by (A.7) and since $\eta^\epsilon(0)$ is ρ_η distributed we have that

$$\mathbb{E}\,|Q^\epsilon(t)|^2 \le \epsilon\, C\,\mathbb{E}(1 + |\eta^\epsilon(t)|^q) \le K\epsilon,$$

for some constants K and C. In particular, a finite dimensional distribution

$$(Q^\epsilon(t_0), Q^\epsilon(t_1), \ldots, Q^\epsilon(t_m))$$

for any $0 \le t_0 < t_1, \ldots, t_m \le T$, will converge in probability to 0 as $\epsilon \to 0$. To show that the family $\{Q^\epsilon(\cdot)\}_{\epsilon>0}$ is tight, we verify the two conditions of Theorem 8.2 of [Billingsley, 2009]. Condition (i) holds immediately since $Q^\epsilon(0) = 0$. Condition (ii) will then follow from Lemma A.0.1 since, given $\delta > 0$ we have that

$$\mathbb{P}\left[\sup_{|t-t_0|<\gamma} |Q^\epsilon(t) - Q^\epsilon(t_0)| > \delta\right] \le 2\delta^{-2}\mathbb{E}\sup_{0\le t\le T}|\epsilon\chi^\epsilon(Y^\epsilon(t),\eta^\epsilon(t))|^2 \to 0,$$

as $\epsilon \to 0$. By Prokhorov's theorem the process $Q^\epsilon(\cdot)$ converges weakly in $C([0,T];\mathbb{R}^d)$ to zero.

Considering the second term of (A.13), since

$$\epsilon\,\mathbb{E}\left|\int_0^t \sqrt{2\Pi\Gamma}\nabla_\eta\chi(Y^\epsilon(s),\eta^\epsilon(s))dW(s)\right|^2$$
$$= 2\epsilon\mathbb{E}\int_0^t \nabla_\eta\chi(Y^\epsilon(s),\eta^\epsilon(s))\,\Gamma\Pi\,\nabla_\eta\chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\,ds$$

by (A.7), it follows that this term converges in distribution to 0. Tightness of this term is a immediate result of the Burkholder–Davis–Gundy inequality, so that this term converges weakly in $C([0,T];\mathbb{R}^d)$ to zero.

Consider the third term of (A.13) term. Define the terms $b_1 : \mathbb{T}^d\times\mathbb{R}^K \to \mathbb{R}^d$ and $b_2 : \mathbb{R}^K \to \mathbb{R}^d$ by

$$b_1(y,\eta) = \mathcal{L}_\eta\chi(y,\eta) - \int_{\mathbb{T}^d}\mathcal{L}_\eta(y,\eta)\chi(y,\eta)\rho_y(dy,\eta),$$

and

$$b_2(\eta) = \int_{\mathbb{T}^d}\mathcal{L}_\eta(y,\eta)\chi(y,\eta)\rho_y(dy,\eta) - \int_{\mathbb{R}^K}\int_{\mathbb{T}^d}\mathcal{L}_\eta(y,\eta)\chi(y,\eta)\rho_y(dy,\eta)\rho_\eta(d\eta).$$

Each component of the function b_1 satisfies the conditions of Proposition A.2.3. Applying this proposition, it follows that the process $t\to\int_0^t b_1(Y^\epsilon(s),\eta^\epsilon(s))\,ds$ converges weakly in $C([0,T];\mathbb{R}^d)$ to 0. Similarly, the term b_2 satisfies the conditions of Lemma A.1.3 so that the stochastic process $t\to\int_0^t b_2(\eta^\epsilon(s))\,ds$ converges weakly in $C([0,T];\mathbb{R}^d)$ to 0. It follows that, as $\epsilon\to 0$, the process

$$t\to\int_0^t \mathcal{L}_\eta\chi(Y^\epsilon(s),\eta^\epsilon(s))\,ds,$$

converges weakly in $C([0,T];\mathbb{R}^d)$ to

$$t\to L\,t,$$

where

$$L = \int_{\mathbb{R}^K}\int_{\mathbb{T}^d}\mathcal{L}_\eta\chi(y,\eta)\,\rho_y(dy,\eta)\rho_\eta(d\eta).$$

Consider the martingale term

$$M^\epsilon(t) = \int_0^t \sqrt{2\Sigma}\left(I+\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s))\right)^\top\,dB(s).$$

The quadratic variation of $M^\epsilon(t)$ is then given by

$$[\![M^\epsilon(t)]\!] = 2\int_0^t \left(I+\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s))\right)g^{-1}\left(I+\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s))\right)^\top\,ds.$$

Applying Lemmas A.2.3 and A.1.3 and arguing as above we see that for $0 \leq t \leq T$:

$$[\![M^\epsilon(t)]\!] \to 2D,$$

in $L^2(\rho_\eta)$, where the diffusion coefficient D is given in (5.18). We can then apply the martingale central limit theorem given in Theorem 7.1.4 of [Ethier and Kurtz, 2009] to show that $M^\epsilon(\cdot)$ converges weakly in $C([0,T];\mathbb{R}^d)$ to the diffusion process $\sqrt{2D}\,B(t)$. Substituting the terms in (A.13) and taking the limit as $\epsilon \to 0$, it follows that $X^\epsilon(\cdot)$ converges weakly in $C([0,T];\mathbb{R}^d)$ to the diffusion process $X^0(\cdot)$ given by the unique weak solution of the Itô SDE

$$dX^0 = L\,dt + \sqrt{2D}dB(t),$$

thus proving the theorem.

A.3 CASE IV

In this section we give a rigorous proof of Theorem 5.3.3, which provides a homogenization result for the Case IV scaling regime, that is, the system described by the system of equations in (5.26). As mentioned in Section 5.3, the fast process $(Y^\epsilon(t), \eta^\epsilon(t))$ has no obvious invariant density, and so we must first establish the existence of an invariant measure ρ under which the fast process is ergodic. We first prove Proposition 5.3.1 which guarantees the existence of a smooth invariant density for the fast process in (5.26) with respect to the Lebesgue density over $\mathbb{T}^d \times \mathbb{R}^K$. This result will be a direct application of [Mattingly and Stuart, 2002, Corollary 2.8], which provides sufficient conditions for an Itô process to be geometrically ergodic with respect to a unique invariant measure. The conditions to be checked are that the process possesses a Lyapunov function V and that the so-called *minorization condition* holds (see Assumption 2.1 of [Mattingly and Stuart, 2002]).

Proof of Proposition 5.3.1. Define $V : \mathbb{T}^d \times \mathbb{R}^K \to [1,\infty)$ as follows

$$V(y,\eta) = 1 + \frac{1}{2}|\eta|^2.$$

It is clear that $\lim\limits_{|(y,\eta)|\to\infty} V(y,\eta) = \infty$. Moreover,

$$\begin{aligned}
\mathcal{G}V(y,\eta) &= \frac{1}{\sqrt{|g|(y,\eta)}}\nabla_y \cdot \left(\sqrt{|g|(y,\eta)}g^{-1}(y,\eta)\nabla_y V(y,\eta)\right) + \mathcal{L}_\eta V(y,\eta)\\
&= \mathcal{L}_\eta V(y,\eta)\\
&\leq -aV(y,\eta) + b,
\end{aligned}$$

where $a = \min\limits_{k\in\mathbb{K}} \Gamma_k > 0$ and $b > 0$ are constants. It follows that V is a Lyapunov function for the fast process $(Y^\epsilon(t), \eta(t))$.

We now wish to verify the two conditions of Assumptions 2.6 of [Mattingly and Stuart, 2002] hold, which together imply the Minorization Condition is satisfied. We first show that the transition kernel $P(t, x, \cdot)$ possesses a smooth density $p(t, y, \eta)$ with respect to the Lebesgue measure on $\mathbb{T}^d \times \mathbb{R}^K$. The process $Z(t) = (Y(t), \eta(t))$ is the unique solution of the $\mathbb{T}^d \times \mathbb{R}^K$ valued-SDE

$$dZ(t) = \hat{F}(Z(t))dt + \hat{\Sigma}(Z(t))dW(t),$$

where

$$\hat{F}((y, \eta)) = (F(y, \eta), -\Gamma\eta)^\top, \tag{A.14}$$

and

$$\hat{\Sigma}((y, \eta)) = \sqrt{2} \begin{pmatrix} \Sigma(y, \eta) & 0 \\ 0 & \Gamma\Pi \end{pmatrix},$$

where $W(t)$ is a standard Brownian motion on $\mathbb{T}^d \times \mathbb{R}^K$. The diffusion coefficient $\hat{\Sigma}(z)$ is clearly non-singular for all z, so that its columns span $\mathbb{T}^d \times \mathbb{R}^K$. By [Rogers and Williams, 2000, Theorem 38.16] it follows that the transition kernel possesses a smooth density $p(t, x, y)$, so that Assumption 2.6(ii) of [Mattingly and Stuart, 2002] holds. To prove part (i) of Assumption 2.6 we now show that in any positive time, any open set in $\mathbb{T}^d \times \mathbb{R}^K$ may be reached with positive probability. Indeed let $t > 0$, $x, z \in \mathbb{T}^d \times \mathbb{R}^K$ and $\delta > 0$. We consider the probability of hitting the ball $B_\delta(y)$ of radius δ centered at z. As in [Mattingly and Stuart, 2002, Lemma 3.4] we consider the control problem derives from (A.14), namely

$$\frac{dQ(t)}{dt} = \hat{F}(Q(t)) + \hat{\Sigma}(Q(t))\frac{dU}{dt}. \tag{A.15}$$

Choose a C^∞ path $Q(t)$ in $\mathbb{T}^d \times \mathbb{R}^d$ such $Z(0) = x$ and $Z(t) = z$. Since $\hat{\Sigma}(\cdot)$ is invertible, this choice of $Q(\cdot)$ uniquely defines a path $U(\cdot)$ which satisfies (A.15). Moreover, $U(\cdot)$ will be as regular as the coefficients of (A.15), thus will be C^∞. Let $W(\cdot)$ be a standard Weiner process on $\mathbb{T}^d \times \mathbb{R}^K$ starting from $W(0) = x$. Then by [Stroock and Karmakar, 1982, Theorem 4.20], for all $\epsilon > 0$ the event

$$\sup_{s \leq t} |W(s) - U(s)| < \epsilon, \tag{A.16}$$

has positive probability. Let $W(\cdot)$ be a particular realisation for which (A.16) holds. Comparing $Q(\cdot)$ with $Z(\cdot)$ we have that,

$$|Z(t) - Q(t)| \leq \int_0^t \mathrm{Lip}\left[\hat{F}\right] |Z(s) - Q(s)| + \left\|\hat{\Sigma}\right\|_\infty \epsilon \, ds,$$

where Lip$[f]$ denotes the *local* Lipshitz constant of f. It follows by Gronwall's inequality that $|Z(t) - Q(t)| = K(t)\epsilon$ for some constant K. It follows from the above construction that for all $\delta > 0$, the probability that $Z(t)$ lies within $B(z, \delta)$ is positive. Given this result it is straightforward to verify that the conditions of Assumption 2.6(i) hold. Applying [Mattingly and Stuart, 2002, Corollary 2.8], the process

$Z(t)$ possesses a unique invariant measure ρ and moreover, there exist constants $\mu \in (0,1)$ and $\kappa > 0$ such that for all measurable $f : \mathbb{T}^d \times \mathbb{R}^K \to \mathbb{R}$ such that $|f| \leq V$, and for all $(y,\eta) \in \mathbb{T}^d \times \mathbb{R}^K$:

$$\left| P(t)f((y,\eta)) - \int f(y,\eta)\, \rho(dy, d\,\eta) \right| \leq \kappa e^{-\mu t} V(y,\eta),$$

where $P_t f(\cdot)$ denotes the contraction semigroup generated by the infinitesimal generator $\mathcal{G}$ of the process $Z(t) = (Y(t), \eta(t))$. Integrating both sides with respect to the invariant measure we have that

$$\left\| P(t)f - \int f(y,\eta)\, \rho(dy, d\,\eta) \right\|_{L^2(\rho)} \leq \kappa e^{-\mu t} \left\| V \right\|_{L^2(\rho)}.$$

However as V is growing at most polynomially in η it follows that $\|V\|_{L^2(\rho)} < \infty$ so that (5.32) holds. Since ρ is an invariant measure it follows that

$$\mathcal{G}^* \rho = 0 \tag{A.17}$$

in the sense of distributions. However, noting that $\mathcal{G}$ is hypoelliptic , it follows by Hormander's theorem that ρ is a C^∞ function and solves equation (A.17) in the strong sense. Thus, the invariant measure ρ possesses a smooth, positive density with respect to the Lebesgue measure on $\mathbb{T}^d \times \mathbb{R}^K$. $\square$

Using the convergence estimates of Proposition 5.3.1 we then show that the Poisson equation

$$-\mathcal{G}\phi(y,\eta) = b(y,\eta), \qquad (y,\eta) \in \mathbb{T}^d \times \mathbb{R}^K,$$

possesses a unique solution $\phi \in L^2(\rho)$ provided the RHS satisfies $\int b(y,\eta)\, d\rho = 0$. In lieu of standard elliptic results, we make use of the fact that the infinitesimal generator is hypoelliptic and use Hörmander's theorem to show that the solution is sufficiently regular.

Proposition A.3.1. *Let $b \in L^2(\rho)$ be a C^∞ function which satisfies (5.29) and suppose the centering condition*

$$\int b(y,\eta)\, \rho(dy, d\,\eta) = 0,$$

holds. Then, there exists a unique, smooth solution $\phi \in D(\mathcal{G})$ in $L^2(\rho)$ to the following Poisson equation

$$-\mathcal{G}\phi = b \text{ in } L^2(\rho), \tag{A.18}$$

which satisfies $\int \phi(y,\eta)\rho(dy, d\eta) = 0$. The solution ϕ satisfies

$$|\phi(y,\eta)| \leq CV(\eta), \qquad (y,\eta) \in \mathbb{T}^d \times \mathbb{R}^K \tag{A.19}$$

where $C > 0$ is a constant independent of (y, η) and $V(\eta)$ is given in (5.30). Moreover,

$$\int \nabla_y \phi \cdot g^{-1} \nabla_y \phi \, d\rho + \int \nabla_\eta \phi \cdot \Gamma \Pi \nabla_\eta \phi \, d\rho = -2 \int \phi \, \mathcal{G} \phi \, d\rho < \infty. \tag{A.20}$$

Proof. Define

$$\phi(x, \eta) = \int_0^\infty P(t) b(x, \eta) \, dt.$$

Estimate (5.32) implies that $\phi \in L^2(\rho)$. Moreover, using the identity

$$P(t)\phi - \phi = \int_0^t P(s) b \, ds,$$

we see that $\lim_{t \to 0} \frac{1}{t} \left(P(t)\phi - \phi \right)$ converges strongly in $L^2(\rho)$ so that $\phi \in \mathcal{D}(\mathcal{G})$ and also ϕ solves $-\mathcal{G}\phi = b$ in the sense of distributions. Invoking Hormander's theorem once again since b is C^∞ then ϕ is also C^∞ and solves (A.18) in the strong sense. The pointwise bound (A.19) follows by applying (5.31) to ϕ to get that for all $(y,\eta_0) \in \mathbb{T}^d \times \mathbb{R}^K$

$$\begin{aligned} |\phi(y_0, \eta_0)| &\leq \int_0^\infty \left| \mathbb{E}^{(y_0, \eta_0)} b(Y(t), \eta(t)) \right| \, dt \leq \int_0^\infty C e^{-\mu t} dt \, V(\eta_0) \\ &= \frac{C}{\mu} V(\eta_0). \end{aligned}$$

Finally, using an argument similar to Proposition 6.12 of [Pavliotis and Stuart, 2008] we have that

$$\begin{aligned} \int \phi \left(-\mathcal{G}\phi \right) d\rho &= \int \phi (-\mathcal{G}_{sym} \phi) \, d\rho \\ &= \int \nabla_y \phi \cdot g^{-1} \nabla_y \phi \, d\rho + \int \nabla_\eta \phi \cdot \Gamma \Pi \, \nabla_\eta \phi \, d\rho, \end{aligned}$$

where $\mathcal{G}_{sym}$ denotes the symmetrization of $\mathcal{G}$. Since both since b and ϕ are in $L^2(\rho)$, this implies the bound (A.20). □

We now provide a proof of Theorem 5.3.3. Having shown the existence of a ergodic invariant measure, as well a smooth solution to the poisson equation (A.18), the proof of the homogenization theorem is now standard, following those in Chapter 3 of [Bensoussan et al., 1978], or [Pardoux, 1999]. The approach taken here closely follows the methodology used in the similar scenario considered in [Cattiaux et al., 2010].

Proof of Theorem 5.3.3. Let χ be the unique solution of the cell equation

$$\mathcal{G}\chi(y, \eta) = -F(y, \eta), \qquad (y, \eta) \in \mathbb{T}^d \times \mathbb{R}^K,$$

satisfying $\int \chi d\rho = 0$, which exists and is smooth by Proposition 5.3.2. Applying Itô's formula for $\chi(Y^\epsilon(t),\eta^\epsilon(t))$, for $0 \le t \le T$,

$$
\begin{aligned}
\chi(Y^\epsilon(t),\eta^\epsilon(t)) = \chi(Y^\epsilon(0),\eta^\epsilon(0))& \\
&+ \frac{1}{\epsilon^2}\int_0^t \mathcal{G}\chi(Y^\epsilon(s),\eta^\epsilon(s))\,ds \\
&+ \int_0^t \sqrt{\frac{1}{\epsilon^2}\Sigma(Y^\epsilon(s),\eta^\epsilon(s))}\nabla_y \chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\,dB(s) \\
&+ \int_0^t \sqrt{\frac{2}{\epsilon^2}\Gamma\Pi}\nabla_\eta \chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\,dW(s).
\end{aligned}
$$

Therefore we can write the SDE for $X^\epsilon(t)$ as follows

$$
\begin{aligned}
X^\epsilon(t) = X^\epsilon(0) &+ \epsilon\left(\chi(Y^\epsilon(0),\eta^\epsilon(0)) - \chi(Y^\epsilon(t),\eta^\epsilon(t))\right) \\
&+ \int_0^t \sqrt{2\Sigma(Y^\epsilon(s),\eta^\epsilon(s))}\left(I + \nabla_y \chi(Y^\epsilon(s),\eta^\epsilon(s))\right)^\top\,dB(s) \\
&+ \int_0^t \sqrt{2\Gamma\Pi}\nabla_\eta \chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\,dW(s).
\end{aligned}
$$

Define $Q^\epsilon(t)$ to be the process $Q^\epsilon(t) = \epsilon\left(\chi(Y^\epsilon(0),\eta^\epsilon(0)) - \chi(Y^\epsilon(t),\eta^\epsilon(t))\right)$. By the smoothness of $\chi(\cdot,\cdot)$, it is clear that the realisations of this process lie in $C([0,T];\mathbb{R}^2)$. Since the process $\eta^\epsilon(0)$ is ρ_η-distributed, and $\chi \in L^2(\rho)$ by Proposition 5.3.2 we have that

$$
\mathbb{E}\left|\epsilon\,\chi(Y^\epsilon(t),\eta^\epsilon(t))\right|^2 \le C^2\mathbb{E}\left|\epsilon V(\eta^\epsilon(t))\right|^2 = C^2\,\epsilon^2\mathbb{E}(1+|\eta^\epsilon(t)|^2)^2 \to 0,
$$

as $\epsilon \to 0$, since the OU process possesses finite moments. In particular, a finite dimensional distribution

$$
\left(Q^\epsilon(t_0), Q^\epsilon(t_1), \ldots, Q^\epsilon(t_m)\right),
$$

for any $0 \le t_0 < t_1 \ldots, t_m \le T$, will converge in distribution to 0 as $\epsilon \to 0$.

We wish to show that the family $\{Q^\epsilon\}_{\epsilon>0}$ is tight. To this end, we wish to confirm the conditions of Theorem 8.2 of [Billingsley, 2009]. Condition (i) follows trivially from the fact that $Q^\epsilon(0) = 0$. As in the previous proofs, Condition (ii) then follows from Lemma A.0.1. It follows that this family is tight, and so, by Prokhorov's theorem the process $Q^\epsilon(\cdot)$ converges weakly in $C([0,T];\mathbb{R}^d)$ as $\epsilon \to 0$.

Consider now the terms

$$
M_1^\epsilon(t) := \int_0^t \sqrt{2\Sigma(Y^\epsilon(s),\eta^\epsilon(s))}\left(I + \nabla_y \chi(Y^\epsilon(s),\eta^\epsilon(s))\right)^\top\,dB(s),
$$

and

$$M_2^\epsilon(t) := \int_0^t \sqrt{2\Gamma\Pi}\nabla_\eta\chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\, dW(s).$$

Computing the quadratic variations of M_1^ϵ and M_2^ϵ,

$$\begin{aligned}
\mathbb{E}[\![M_1^\epsilon]\!](t) &= 2\mathbb{E}\int_0^t \left(I+\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s)\right)\Sigma(Y^\epsilon(s),\eta^\epsilon(s))\left(I+\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s))\right)^\top\, ds\\
&\le 2\int_0^t \mathbb{E}\left(\Sigma(Y^\epsilon(s),\eta^\epsilon(s))\right)\, ds\\
&+\ 2\int_0^t \mathbb{E}\left(\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s))\Sigma(Y^\epsilon(s),\eta^\epsilon(s))\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\right)\, ds\\
&+4\int_0^t \mathbb{E}\left(\Sigma(Y^\epsilon(s),\eta^\epsilon(s))\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s))^\top\right)\, ds.
\end{aligned}$$

Since the initial value of the fast process $(Y^\epsilon(0),\eta^\epsilon(0))$ is ρ-distributed we obtain, applying the Cauchy-Schwartz inequality for the last line:

$$\begin{aligned}
&\mathbb{E}\mid [\![M_1^\epsilon]\!](t)|_F\\
&\le 2\mathbb{E}\,\mathrm{tr}\int_0^{\mathrm{t}} \left(\mathrm{I}+\nabla_{\mathrm{y}}\chi(\mathrm{Y}^\epsilon(\mathrm{s}),\eta^\epsilon(\mathrm{s})\right)\Sigma(\mathrm{Y}^\epsilon(\mathrm{s}),\eta^\epsilon(\mathrm{s}))\left(\mathrm{I}+\nabla_{\mathrm{y}}\chi(\mathrm{Y}^\epsilon(\mathrm{s}),\eta^\epsilon(\mathrm{s}))\right)^\top\,\mathrm{ds}\\
&\le 2t\left(2+\mathrm{tr}\int\nabla_{\mathrm{y}}\chi\mathrm{g}^{-1}\nabla_{\mathrm{y}}\chi^\top\,\rho(\mathrm{dy},\mathrm{d}\eta)+\left(4\,\mathrm{tr}\int\nabla_{\mathrm{y}}\chi\mathrm{g}^{-1}\nabla_{\mathrm{y}}\chi^\top\,\rho(\mathrm{dy},\mathrm{d}\eta)\right)^{\frac{1}{2}}\right),
\end{aligned}$$

which is finite by relation (5.35). Similarly

$$\frac{1}{2}\,\mathbb{E}\,|\,[\![M_2^\epsilon]\!](t)|_F \le t\left(\mathrm{tr}\int\nabla_\eta\chi\,\Gamma\Pi\,\nabla_\eta^\top\chi\,\rho(\mathrm{dy},\mathrm{d}\eta)\right),$$

which is also finite by (5.35). It follows that M_1^ϵ and M_2^ϵ are $L^2(\rho)$ martingales. Moreover, by the ergodicity of the fast process, taking the limit as $\epsilon\to 0$.

$$\begin{aligned}
&\frac{1}{2}\,[\![M_1^\epsilon]\!](t)\\
&=\int_0^t \left(I+\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s)\right)\Sigma(Y^\epsilon(s),\eta^\epsilon(s))\left(I+\nabla_y\chi(Y^\epsilon(s),\eta^\epsilon(s))\right)^\top\, ds\\
&=\int_0^t \left(I+\nabla_y\chi(Y(s/\epsilon^2),\eta(s/\epsilon^2)\right)\Sigma(Y(s/\epsilon^2),\eta(s/\epsilon^2))\left(I+\nabla_y\chi(Y(s/\epsilon^2),\eta(s/\epsilon^2))\right)^\top\, ds\\
&\to t\int\left(I+\nabla_y\chi\right)g^{-1}\left(I+\nabla_y\chi\right)^\top\,\rho(dy,d\eta).
\end{aligned}$$

Similarly as $\epsilon\to 0$,

$$\frac{1}{2}[\![M_2^\epsilon]\!](t)\to t\int\nabla_\eta\chi\Gamma\Pi\nabla_\eta\chi^\top\,\rho(dy,d\eta),$$

We can now apply the martingale central limit theorem given in Theorem 7.1.4 of [Ethier and Kurtz, 2009] to show that $(M_1^\epsilon + M_2^\epsilon)(\cdot)$ converges weakly in the Skorohod topology to the diffusion process $X^0(\cdot)$ respectively, where

$$X^0(t) = \sqrt{2D} dB(t),$$

where the effective diffusion term is given in (5.39), thus proving the desired result.

$\square$

Appendix B

ADDITIONAL RESULTS

For the sake of completeness, we provide a proof of the Nash inequality for $\mathbb{T}^d$. While the standard Nash-inequality on $\mathbb{R}^d$ is widely known (see for example, the original paper [Nash, 1958]), corresponding results for bounded domains are less well known. We note, in particular, the introduction of the additional $\|u\|^2_{L^2(\mathbb{T}^d)}$ term that arises in the equality due to the zero Fourier mode. While this is not the "tightest" form of the result (as will be apparent from the proof), it is of the required form to apply the heat kernel estimates of Bakry et al. [2010].

Theorem B.0.2. *There exists a constant $C > 0$ such that for all $u \in H^1(\mathbb{T}^d)$,*

$$\|u\|^{1+d/2}_{L^2(\mathbb{T}^d)} \leq C \|u\|_{L^1(\mathbb{T}^d)} \left(\|u\|^2_{L^2(\mathbb{T}^d)} + \|\nabla u\|^2_{L^2(\mathbb{T}^d)} \right)^{d/4}. \tag{B.1}$$

Proof. We use an argument similar to the proof of the standard Nash inequality but use Fourier series in the place of Fourier transforms. Let $r > 0$ and suppose

$$u = \sum_{k \in \mathbb{Z}^d} u_k e_k,$$

where $\{e_k\}_{k \in \mathbb{Z}^d}$ is the standard fourier basis for $L^2(\mathbb{T}^d)$. Then by Parseval's identity

$$\|u\|^2_{L^2(\mathbb{T}^d)} = \sum_{|k| \leq r} u_k^2 + \sum_{|k| > r} u_k^2.$$

Considering the second term on the right hand side

$$\sum_{|k|>r} u_k^2 \leq \sum_{|k|>r} \frac{k^2}{r^2} u_k^2 \leq \left(\frac{1}{2\pi\, r} \right)^2 \|\nabla u\|^2_{L^2(\mathbb{T}^d)}.$$

Considering the first term we have that

$$\sum_{|k| \leq r} u_k^2 \leq \sum_{|k| \leq r} \|u\|^2_{L^1(\mathbb{T}^d)} \leq \Big((2r)^d + 1 \Big) \|u\|^2_{L^1(\mathbb{T}^d)}.$$

Therefore for some constant $K > 0$, the following estimate holds for all $u \in H^1(\mathbb{T}^d)$ and all $r > 0$,

$$\|u\|_{L^2(\mathbb{T}^d)}^2 \leq K \left[r^{-2} \|\nabla u\|_{L^2(\mathbb{T}^d)}^2 + (r^d + 1) \|u\|_{L^1(\mathbb{T}^d)}^2 \right].$$

The optimal value of r is given by

$$r = \left(\frac{2}{d} \frac{\|\nabla u\|_{L^2(\mathbb{T}^d)}^2}{\|u\|_{L^1(\mathbb{T}^d)}^2} \right)^{1/(d+2)},$$

so that for some constant C depending on d alone

$$\begin{aligned}
\|u\|_{L^2(\mathbb{T}^d)}^2 &\leq C \left[\|u\|_{L^1(\mathbb{T}^d)}^2 + \|u\|_{L^1(\mathbb{T}^d)}^{4/(d+2)} \|\nabla u\|_{L^2(\mathbb{T}^d)}^{2d/(d+2)} \right] \\
&= C \|u\|_{L^1(\mathbb{T}^d)}^{4/(d+2)} \left[\|u\|_{L^1(\mathbb{T}^d)}^{2d/(d+2)} + \|\nabla u\|_{L^2(\mathbb{T}^d)}^{2d/(d+2)} \right] \\
&\leq C \|u\|_{L^1(\mathbb{T}^d)}^{4/(d+2)} \left[\|u\|_{L^2(\mathbb{T}^d)}^2 + \|\nabla u\|_{L^2(\mathbb{T}^d)}^2 \right]^{d/(d+2)},
\end{aligned}$$

where we use the fact that $\|u\|_{L^1(\mathbb{T}^d)} \leq \|u\|_{L^2(\mathbb{T}^d)}$. This implies that

$$\|u\|_{L^2(\mathbb{T}^d)}^{1+2/d} \leq C \|u\|_{L^1(\mathbb{T}^d)} \left[\|u\|_{L^2(\mathbb{T}^d)}^2 + \|\nabla u\|_{L^2(\mathbb{T}^d)}^2 \right]^{4/d}.$$

□

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